Sections 1.1–1.4
Review

§ 1.4 + Review: (Basic Facts about Matrices)

Problem 1: How to program computer to solve systems of linear equations ~ Gaussian Elimination

Ex

$$2u + v + w = 5$$
$$4u - 6v = -2$$
$$-2u + 7v + 2w = 9$$

linear because every unknown appears at most multi'd by const

$$\Leftrightarrow \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix} \qquad (1)$$

$$A \cdot x = b$$

$$x = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \quad b = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix} \quad A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

- (1) tells how matrix multiplication is defined: eg

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -8 \\ 10 \end{bmatrix} \quad \text{etc}$$

- For multiplication Ax, we can view this two ways

1. By how x acts on the rows of A
2. By how x acts on the colms of A

The interplay betw the two ways of looking at this give unexpected results (even deep results!)

1. We can view x as acting on the rows of A:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} = \begin{bmatrix} - & r_1 & - \\ - & r_2 & - \\ - & r_3 & - \end{bmatrix} \qquad \begin{aligned} r_1 &= (2,1,1) \\ r_2 &= (4,-6,0) \\ r_3 &= (-2,7,2) \end{aligned}$$

$$x = \begin{bmatrix} 1 \\ c \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

<u>Then</u>: $$Ax = \begin{bmatrix} - & r_1 & - \\ - & r_2 & - \\ - & r_3 & - \end{bmatrix} \begin{bmatrix} 1 \\ c \\ 1 \end{bmatrix} = \begin{bmatrix} r_1 \cdot c \\ r_2 \cdot c \\ r_3 \cdot c \end{bmatrix}$$

"The k'th row of the product is the k'th row of A dotted with c"

Recall: eg $r_1 \cdot c = (2,1,1)\cdot(1,2,-1)$

$$= 2+2-1 = 3$$

In general: $x = (x_1, \dots, x_n)$ $y = (y_1, \dots, y_n)$

$$x \cdot y = x_1 y_1 + \dots + x_n y_n$$

$$= \|x\| \|y\| \cos\theta$$

Thus: $x \cdot y = 0$ iff $x \perp y$

Application ①: $Ax = 0$ iff $x \perp$ the rows of A

Ex solve $\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0$

$$A \quad x \quad = 0$$

$x \perp (2,1) \Rightarrow x = t(-1,2)$ any multiple t

Application 2 Solve $\begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 1 \\ 3 & 0 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = 0$

$A \quad x = 0$

$r_3 = r_1 + r_2$ so ~~the~~ $Ax = 0$ iff x is $\perp$ plane spanned by r_1 & r_2:

$$n = r_1 \times r_2 = \begin{vmatrix} i & j & k \\ 2 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = i(2) - j(1) + k(-3)$$

$x = t(2, -1, -3)$

any scalar multiple

check: $\begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 1 \\ 3 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2t \\ -t \\ -3t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ ✓

2. We can view x as acting on colms of A:

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} \cdot u + \begin{bmatrix} 1 \\ -6 \\ 7 \end{bmatrix} v + \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} w$$

"In the product Ax, we can view x as giving the linear combination of the colms of A that makes the colm of Ax"

"Ax = the linear combination of the colms of A determined by x"

We can picture this:

$$A = \begin{bmatrix} | & | & | \\ c_1 & c_2 & c_3 \\ | & | & | \end{bmatrix} \qquad c_i \equiv i\text{-column of } A$$

$$Ax = \begin{bmatrix} | \\ c_1 \\ | \end{bmatrix} u + \begin{bmatrix} | \\ c_2 \\ | \end{bmatrix} v + \begin{bmatrix} | \\ c_3 \\ | \end{bmatrix} w$$

<u>Application</u> 3 We can only solve

$$Ax = b$$

if b is in the span of the colums of A.

<u>Defn</u>: We say b is in the span of $\{r_1, \ldots, r_n\}$ if $\exists$ constants $\alpha_1, \ldots, \alpha_n$ such that $b = \alpha_1 r_1 + \cdots + \alpha_n r_n$.

$$\mathbb{R}^3 = \{(u,v,w) : u,v,w \in \mathbb{R}\}$$

$$\mathbb{R}^n = \{(x_1, \ldots, x_n) : x_1, \ldots, x_n \in \mathbb{R}\}$$

Ex: For what b does $Ax=b$ have a soln?

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

Ans: $Ax = \begin{bmatrix} 1 \\ 2 \end{bmatrix} u + \begin{bmatrix} 1 \\ 2 \end{bmatrix} v = \begin{bmatrix} 1 \\ 2 \end{bmatrix} (u+v) = b$

$\Rightarrow$ b must be a scalar multiple of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$b = t(1,2)$$

Ex: For what b does $Ax=b$ have a soln?

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & 2 \\ -1 & 1 & 0 \end{bmatrix}$$

sum of 1st two colms

Ans $Ax = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} v + \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} w$

$$= \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} (u+w) + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} (v+w)$$

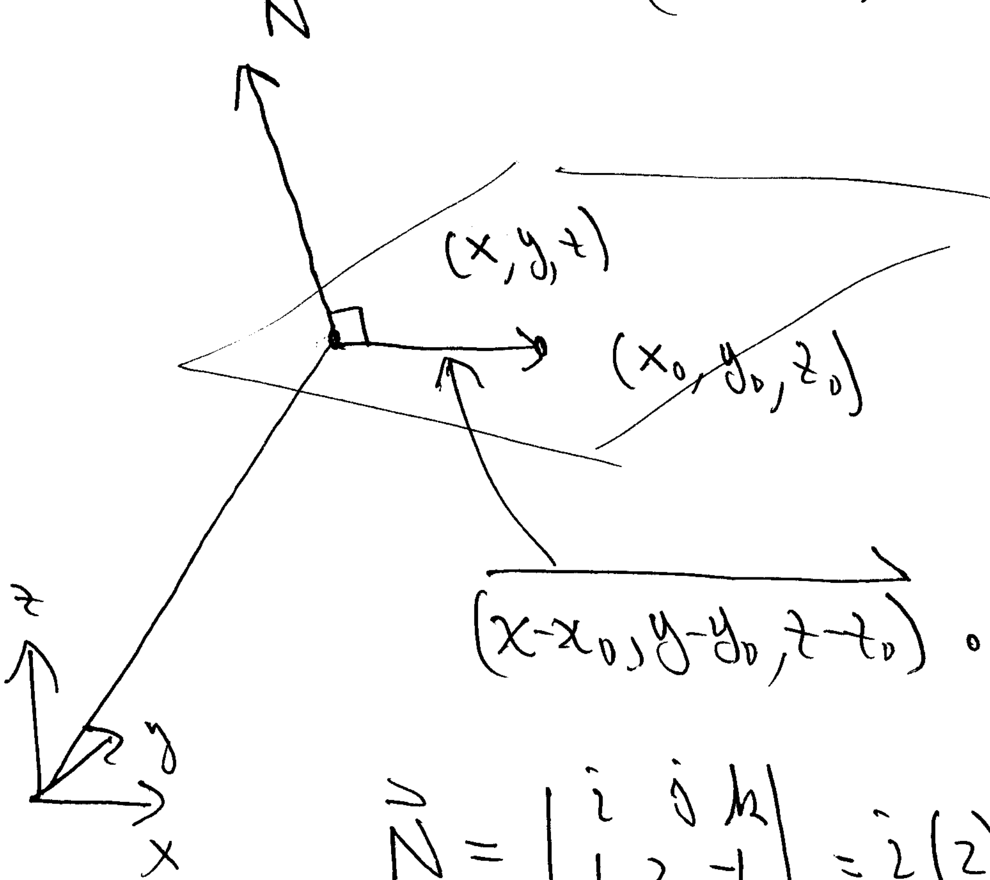
$\Rightarrow$ b must be in the plane spanned by a lin comb of $(1,2,-1)$ & $(1,0,1)$

Eqn of plane: $A(x-x_0)+B(y-y_0)+C(z-z_0)=0$

$\vec{N}=(A,B,C)$ normal to plane

(x_0,y_0,z_0) pt on plane

$=0$ in our case



$\overrightarrow{(x-x_0,\,y-y_0,\,z-z_0)}\cdot\vec{N}=0$

$\vec{N}=\begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ 1 & 0 & 1 \end{vmatrix}=\underset{\sim}{i}(2)-\underset{\sim}{j}(2)+\underset{\sim}{k}(-2)$

$b=(x,y,z)$ must satisfy $2x-2y-2z=0$

- More generally we can define the product of two matrices:

$$\underset{n\times k}{A} \cdot \underset{k\times n}{B}$$

$$B = \begin{bmatrix} | & & | \\ c_1 & \cdots & c_n \\ | & & | \end{bmatrix} \qquad A = \begin{bmatrix} - & r_1 & - \\ & \vdots & \\ - & r_m & - \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} | & | & & | \\ Ac_1 & Ac_2 & \cdots & Ac_n \\ | & | & & | \end{bmatrix} = (r_i \cdot c_j)$$

A acts colm by colm

the (i,j)th entry of AB is $r_i \cdot c_j$

Ex:

$$\begin{bmatrix} 2 & 1 \\ -1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 & -1 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 2 & -1 \\ -1 & -2 & -1 & 1 \\ 0 & 2 & 1 & -2 \end{bmatrix}$$

$3 \times 2 \qquad 2 \times 4$

Formula: $A = (a_{ij}) \quad B = (b_{ij})$

$$AB = \left(\sum_{\sigma=1}^{k} a_{i\sigma} b_{\sigma j} \right) = \left(r_i \cdot c_j \right)$$

the sum gives the dot product

Conclude: If we view matrix mult AB as the colms of B acting on rows of A then write

$$AB = \text{"}(r_i \cdot c_j)\text{"} = AB$$

Appl: if all $c_j \perp r_i \Rightarrow AB = 0$ ✓