§1.5 ③ Gaussian Elimination

☒ Gaussian Elimination: Method for solving

$$\underset{(n\times n)}{A}\underset{(n\times 1)}{x}=\underset{(n\times 1)}{b}$$

when $\exists!$ solution:

- Assume A $n\times n$ & consider $Ax=b$

$$\begin{aligned} 2u+v+w &= 5 \\ 4u-6v &= -2 \\ -2u+7v+2w &= 9 \end{aligned} \qquad (1)$$

$\Leftrightarrow$

$$\underset{A}{\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}}\underset{x}{\begin{bmatrix} u \\ v \\ w \end{bmatrix}} = \underset{b}{\begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}} \qquad (2)$$

<u>Idea</u>: "You can add a mult of one eqvn to another & not change soln space"

Q1: when does $Ax=b$ have a unique soln?

Thm① $Ax=b$ has a ! soln iff A^{-1} exists:

A^{-1} satisfies $A^{-1}A = I = AA^{-1}$

∃b!

Pf. $Ax=b$
$A^{-1}Ax = A^{-1}b$
$x = A^{-1}b$ ✓

Thm② A^{-1} exists iff $\det A \neq 0$

Thm③ $\det(A\cdot B) = |A||B|$

Defn $|A| = \det|A|$: By induction on size of matrix

(2×2) $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc$

(3×3) expand about any row or colm:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

(n×n) $A = (a_{ij})$

$|A_{ij}|$ = det of the $(n-1)\times(n-1)$ matrix obtained by deleting the ith row & jth colm

$$\begin{vmatrix} & - & \\ a_{i1} & a_{i2} \cdots & a_{in} \\ & - & \end{vmatrix} = (-1)^{i+1} a_{i1}|A_{i1}| + (-1)^{i+2} a_{i2}|A_{i2}| + \cdots + (-1)^{i+n} a_{in}|A_{in}|$$

<u>Check</u> $\begin{vmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{vmatrix} = (-1)^{1+2} 4 \begin{vmatrix} 1 & 1 \\ 7 & 2 \end{vmatrix} + (-1)^{2+2}(-6)\begin{vmatrix} 2 & 1 \\ -2 & 2 \end{vmatrix} + 0$

$= -4(2-7) - 6(4+2) = 20 - 36 = -16 \neq 0$

$\therefore$ (2) has a unique soln

- To solve (2) it is more efficient to use Gaussian Elimination than to find A^{-1}

subt $2 \times$ Eq1 from Eq2

subt $(-1) \times$ Eq1 from Eq3

$$\begin{aligned} 2u + v + w &= 5 \\ 4u - 6v &= -2 \\ -2u + 7v + 2w &= 9 \end{aligned} \Rightarrow \begin{aligned} 2u + v + w &= 5 \\ -8v - 2w &= -12 \\ 8v + 3w &= 14 \end{aligned} \text{ add}$$

$$\Rightarrow \begin{aligned} 2u + v + w &= 5 \\ -8v - 2w &= -12 \\ w &= 2 \end{aligned} \Rightarrow \begin{aligned} 2u + 1 + 2 &= 5 \Rightarrow \\ -8v - 2(2) &= -12 \Rightarrow v = 1 \\ w &= 2 \end{aligned}$$

upper triangular with "pivots" on diagonal

We do this systematically keeping track of pivots & multipliers...

Row reduce matrix by <u>elementary matrices</u>

<u>Defn</u>: $E_{ij}(a)$ = "identity matrix with a in the (i,j) entry"

always $i \neq j$

<u>Eg</u> $n=3$: $E_{21}(-2) = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_{32}(2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$

<u>Note</u> ①: If $i > j$, $E_{ij}(a)$ is <u>lower</u> <u>triangular</u> (0's above the diagonal) with 1's on diag.

<u>Note</u> ②: If $i < j$, $E_{ij}(a)$ is <u>upper</u> <u>triangular</u> (0's below diag) with 1's on diag

Eg $n=4$ $E_{23}(-1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

<u>Theorem</u>: Let A be $n \times n$. Then

$E_{ij}(a) \cdot A$ = "the matrix obtained from A by adding a times row j to row i"

$A \cdot E_{ij}(a)$ = "the matrix obtained from A by adding a times colm i to colm j"

"Pf by example" $A = \begin{bmatrix} - & r_1 & - \\ - & r_2 & - \\ - & r_3 & - \end{bmatrix} = \begin{bmatrix} | & | & | \\ c_1 & c_2 & c_3 \\ | & | & | \end{bmatrix}$

$$\underset{E_{21}(a)}{\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} - & r_1 & - \\ - & r_2 & - \\ - & r_3 & - \end{bmatrix}} = \begin{bmatrix} - & r_1 & - \\ - & ar_1 + r_2 & - \\ - & r_3 & - \end{bmatrix} \qquad \underset{A}{\begin{bmatrix} | & | & | \\ c_1 & c_2 & c_3 \\ | & | & | \end{bmatrix}} \underset{E_{21}(a)}{\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \begin{bmatrix} | & | & | \\ c_1 + ac_2 & c_2 & c_3 \\ | & | & | \end{bmatrix}$$

<u>Thm</u>: $E_{ij}(a)^{-1} = E_{ij}(-a)$

"pf": $E_{ij}(-a)\, E_{ij}(a) \cdot I = I$ because $E_{ij}(-a)$ undoes what $E_{ij}(a)$ does —

adds $a \times$ row j to row i

subtracts $+a \times$ row j from row i

Pivots and Multipliers in Gaussian Elimination:

$\boxed{\S 1.5}$

__Ex__

$$\begin{aligned} 2u + v + w &= 5 \\ 4u - 6v \quad &= -2 \\ -2u + 7v + 2w &= 9 \end{aligned}$$

$$\begin{array}{c} (\text{mult}) \\ (\ell_{21}=2) \\ (\ell_{31}=-1) \end{array}
\begin{array}{c} (\text{Pivot}) \\ \begin{bmatrix} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{bmatrix} \end{array}
\rightarrow
\begin{array}{c} \\ \\ (\ell_{32}=-1) \end{array}
\begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 8 & 3 & 14 \end{bmatrix}
\rightarrow
\begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\Leftrightarrow \quad \begin{aligned} 2u + v + w &= 5 \\ -8v - 2w &= -12 \end{aligned}$$

$\boxed{w = 2}$

$-8v - 4 = -12$

$\boxed{v = 1}$

$2u + 1 + 2 = 5$

$\boxed{u = 1}$

__Pivots:__ $d_1 = 2,\ d_2 = -8,\ d_3 = 1$

__mult's:__ $\ell_{21} = 2,\ \ell_{31} = -1,\ \ell_{23} = -1$

row ↑ ↖ colm

$$\equiv \begin{bmatrix} \cancel{\ast} & \cancel{\ast} & \cancel{\ast} \\ 2 & \cancel{\ast} & \cancel{\ast} \\ -1 & -1 & \cancel{\ast} \end{bmatrix}$$

- Alternatively write: $Ax = b$: $\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$

- Row reduce A by matrices:

$$\begin{array}{ccccc} U & G = E_{32}(1) & F = E_{31}(1) & E = E_{21}(-2) & A \end{array}$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ -2 & 7 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 8 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_{32}(1)\,E_{31}(1)\,E_{21}(-2)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} = L^{-1}$$

(Note: l_{ij} don't appear naively here)

- $(GFE)A = U \Rightarrow A = (GFE)^{-1}U$

$(GFE)^{-1} = E^{-1}F^{-1}G^{-1} = L$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

always get 1's along diag because $\begin{bmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{bmatrix}$

d_i appear along diagonal!

Conclude: $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

ℓ_{ij} appear below diag!

- Now solve $Ax = b$: $\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$

$A = LU$ so

$$Ax = b \iff LUx = b \iff \boxed{Ux = L^{-1}b}$$

$$\underset{U}{\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}} \underset{x}{\begin{bmatrix} u \\ v \\ w \end{bmatrix}} = \underset{G}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}} \underset{E}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}} \underset{F}{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

In reverse order they just drop in but not in orig order!

$$= \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -12 \\ 2 \end{bmatrix} \Rightarrow \begin{aligned} u &= 1 \\ v &= 1 \\ w &= 2 \end{aligned} \quad \checkmark$$

- Conclude: $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}} \underset{D}{\begin{bmatrix} 2 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & 1/2 & 1/2 \\ 0 & 1 & -2/-8 \\ 0 & 0 & 1/1 \end{bmatrix}}$$

new U always has 1's along diag

<u>General</u>: $A = LDU$

$D = \text{diag}(d_i)$; L has lower diag l_{ij}

L, U have 1's along diag –

2 problems – A not invertible
Rows out of order for elimination