Conclude: When Gaussian Elimination works:

$$Ax = b \qquad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$E_{32}(-l_{32})E_{31}(-l_{31})E_{21}(-l_{21})A = U$$

$$A = \underbrace{E_{21}(l_{21})E_{31}(l_{31})E_{32}(l_{32})}_{L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}} U \qquad \begin{bmatrix} d_1 & * & * \\ 0 & d_2 & * \\ 0 & 0 & d_3 \end{bmatrix}$$

or

$$A = \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}} \underset{D}{\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}}$$

Note: "The order of the E's on RHS makes it so l_{ij} just falls in (i,j)th spot of L"

Eliminate a 4x4 matrix:

order of elimination? $\rightarrow$ $\begin{bmatrix} * & & & \\ 1 & * & & \\ 2 & 4 & * & \\ 3 & 5 & 6 & * \end{bmatrix}$

$U =$ $\quad U = E_{43} E_{42} E_{32} E_{41} E_{31} E_{21} A$

$$\underbrace{E_{21} E_{31} E_{41} E_{32} E_{42} E_{43}}_{I + (l_{ij})} U = A$$

"colm indices non-increasing from right to left"

"row indices unequal when colm indices are the same"

Enough to imply $L = I + (l_{ij})$

__Thm__: $E_{i_n j_n}(l_{i_n j_n}) \cdots E_{i_1 j_1}(l_{i_1 j_1}) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ & 1 & & & 0 \\ & & 1 & & \\ (l_{ij}) & & & \ddots & \\ & & & & 1 \end{bmatrix}$

so long as j_k is non-increasing, $j_{k+1} \le j_k$, and $i_k \ne i_l$ when $j_k = j_l$.

Also: when $j_k = j_l$ and $i_k \ne i_l$, the matrices commute, ie

$$E_{ij}(l_{ij}) E_{i'j}(l_{i'j}) = E_{i'j}(l_{i'j}) \cdot E_{ij}(l_{ij})$$

(See discussion on webpage)

__Cor__: When you do Gaussian Elim you end with

$$A = LU, \quad L = \underbrace{E_{i_n j_n}(l_{i_n j_n}) \cdots E_{i_1 j_1}(l_{i_1 j_1})}_{\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ l_{ij} & & 1 \end{bmatrix}}$$

☒ Conclude: To solve $Ax=b$ most efficiently.

① Find U and $L = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ \ell_{ij} & & 1 \end{bmatrix} \Rightarrow A = LU$

$$Ax=b \Leftrightarrow LUx=b \Leftrightarrow Ux = L^{-1}b = c$$

② Solve $\boxed{Lc=b}$

③ Solve $\boxed{Ux=c}$ "back subst"

The point – you only need L & U, but never need compute L^{-1} or U^{-1} since back-substitution is more efficient.

I.e. $U \cdot Lc=b$

$$\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ \ell_{ij} & & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \leftarrow c_1 = b_1 \quad \vdots \quad \text{etc}$$

$Ux=c$

$$\begin{bmatrix} d_1 & & u_{ij} \\ & \ddots & \\ 0 & & d_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \quad \text{etc} \quad \vdots \quad \leftarrow x_n = \frac{c_n}{d_n}$$

Count the # of arithmetical operations required for Gaussian Elimination

$$\begin{array}{cc} \ell_{21} \longrightarrow \\ \ell_{31} \longrightarrow \\ \vdots \\ \ell_{n1} \longrightarrow \end{array} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ a_{31} & a_{32} & & a_{3n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix}$$

- Start with 1st pivot: $d_1 = a_{11}$
- To get zeros under d_1, each row requires n operations: $\ell_{21} = \frac{a_{21}}{a_{11}}$ = 1-division

  $(n-1)$ - "mult by ℓ_{21} & subtract" operations on rest of the row

- $\exists$ $(n-1)$-rows under 1st pivot $\Rightarrow$

  $n(n-1)$ operations for 1st pivot

- 2nd pivot same except for $(n-1)\times(n-1)$ matrix

$\therefore$ Total operations for row reduction:

$$(n-1)n + (n-2)(n-1) + (n-3)(n-2) + \cdots + 1\cdot 2$$

$$= \sum_{k=1}^{n} (n-1)n = \sum_{k=1}^{n} n^2 - \sum_{k=1}^{n} n = \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}$$

check by induction $\rightarrow$

$$\frac{n(n+1)}{2}\frac{(2n+1)}{3} \qquad \frac{n(n+1)}{2} \qquad = \frac{n(n+1)}{2}\left(\frac{2n+1}{3} - 1\right)$$

$$= \frac{n(n+1)}{2}\left(\frac{2n+1-3}{3}\right)$$

For large n we get $= \dfrac{n(n+1)(n-1)\cancel{2}}{\cancel{2}\cdot 3} = \dfrac{n(n^2-1)}{3}$

$$\# \text{operations} \approx \frac{n^3-n}{3} \approx \frac{n^3}{3} \quad \text{large } n$$

☐ Count # of arithmetic operations for back substitution $Ux = c$ or $Lc = b$:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \\ \vdots & 0 & & \\ 0 & & 0 & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ \vdots \\ c_n \end{bmatrix}$$

(n-1) mult-subtr op's

n-operations $a_{11}x_1 + a_{12}x_2 + \cdots \quad a_{1n}x_n = c_1$

one divide

$\vdots$

one mult-subtr operation

2 operations $\quad a_{n-1\,n-1}x_{n-1} + a_{n-1\,n}x_n = c_{n-1}$

1-division $\quad a_{nn}x_n = c_n$

Conclude: $1 + 2 + \cdots + n = \frac{n(n+1)}{2} \approx \frac{n^2+n}{2}$ operations

"Much faster than Gaussian Elim"

$\boxed{\S 1.6}$ Comments:

- Operation count for computing A^{-1} by Gauss Jordan is n^3

More efficient to compute L & U & solve $Lc=b$ & $Ux=c$ by back subst. since computing L, U is $\frac{n^3}{3}$ & backsubst is $\sim n^2$ ✓

"I.e. computing A^{-1} $[A\,;\,I]$

$$\downarrow \ 2\times \text{ Gaussian Elim}$$

$$[U\,;\,L^{-1}]$$

$$\downarrow \ 2\times \text{ Gaussian Elim}$$

$$[I\,;\,A^{-1}]$$"

$\boxed{\text{Actual count} \simeq n^3 \quad \text{See book}}$

17 B

A symmetric $A = A^T$, $a_{ij} = a_{ji}$ Eg $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 3 \\ -1 & 3 & -1 \end{bmatrix}$

Eg: given any $A_{n \times m}$

- Thm: $\underset{m\times n \; n\times m}{AA^T}$ & $\underset{n\times m \; m\times n}{A^TA}$ are symmetric

$m \times m$ $\quad$ $n \times n$

Pf $(AA^T)^T = (A^{T^T} A^T) = AA^T$ etc

- Thm $A = A^T$ $n\times n$ $\Rightarrow$ if $A = LDU$, then $U = L^T$ (Nice!)

Ie. $A = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ l_{ij} & & 1 \end{bmatrix} \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix} \begin{bmatrix} 1 & & l_{ij} \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

Pf: LDU decomp of A is unique if it exists.

$\therefore$ if $A = LDU$, then $A^T = U^T D L^T = A$

$\therefore$ $U^T = L$ & $L^T = U$ ✓

- $|A| = |LDU| = |L||D||U|$

<u>Note</u>: $|L| = 1 = |U|$

Pf.
$$\begin{vmatrix} 1 & * & * & \cdots \\ 0 & 1 & * & \cdots \\ 0 & 0 & 1 & \\ & & & \ddots & 1 \end{vmatrix}_{n\times n} = 1 \cdot \begin{vmatrix} 1 & * & * & \cdots \\ 0 & 1 & * & \cdots \\ & 0 & \ddots & \\ & & & 1 \end{vmatrix}_{(n-1)(n-1)} = 1\cdot 1 \cdot |L|_{n-2}$$

expand by 1st colm

$$= \cdots 1\cdot 1 \cdots 1 \begin{vmatrix} 1 & * \\ 0 & 1 \end{vmatrix} = 1 \checkmark$$

$\therefore |A| = d_1 \cdot d_2 \cdots d_n$

Conclude "Det A = prod of pivots"

<u><u>Cor</u></u>: $|A| \neq 0$ iff $d_i \neq 0 \ \forall i$.

- Sometimes $|A| \neq 0 \Rightarrow Ax = b$ has ! soln but Elimination algorthm breaks down —

<u>Ex</u>: Say we start with $A_{4\times 4}$ & eliminate to get:

$$\underbrace{P_{34}\, E_{42}(e)\, E_{32}(d)\, E_{41}(c)\, E_{31}(b)\, E_{21}(a)\, A}_{\begin{bmatrix} 2 & 1 & -1 & 2 \\ 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}} = U \quad (*)$$

← need to interchange last two rows to get pivot

$$P_{34} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$ = "matrix obtained from I by interchanging last two rows"

$$P_{34} \begin{bmatrix} - r_1 - \\ - r_2 - \\ - r_3 - \\ - r_4 - \end{bmatrix} = \begin{bmatrix} - r_1 - \\ - r_2 - \\ - r_4 - \\ - r_3 - \end{bmatrix}$$

<u>Problem</u>: cant write $A = LU$!

In this case: Use

<u>Thm</u>: For every A st $|A| \neq 0$ $\exists$ permutation matrix P st Gaussian Elimination works

i.e st
$$PA = LDU$$

<u>Thm</u> (HW): if $A = L_1 D_1 U_1 = L_2 D_2 U_2$
then $L_1 = L_2$, $D_1 = D_2$, $U_1 = U_2$ $\Rightarrow$ <u>unique</u>

<u>Problem</u>: How to find P "ahead of time"?

(A) Guess
(B) As follows —

(B) <u>Thm</u>: $P_{34} \cdot E_{42} = E_{32} \cdot P_{34}$

chng 3 to 4 or 4 to 3
and leave others unchnged

need 4 > 2 in colms

From (*) :

$$\text{LHS } (*) = E_{32}(e)\, E_{42}(d)\, E_{31}(c)\, E_{41}(b)\, E_{21}(a)\, P_{34} A = U$$

"switch $3 \leftrightarrow 4$ as you pass P_{34} in"

Now use "matrices with same j commute"

$$= E_{42}(d)\, E_{32}(e)\, E_{41}(b)\, E_{31}(c)\, E_{21}(a)\, P_{34} A = U$$

$\ell_{42} = -d$ $\quad \ell_{32} = -e$ $\quad \ell_{41} = -b$ $\quad \ell_{31} = -c$ $\quad \ell_{21} = -a$

$$\Rightarrow P_{34} A = LU \text{ with } L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -a & 1 & 0 & 0 \\ -c & -e & 1 & 0 \\ -b & -d & 0 & 1 \end{bmatrix}$$

✓

☒ Permutation matrices:

Thm: $I = \begin{bmatrix} - e_1 - \\ \vdots \\ - e_n - \end{bmatrix}$ $\quad e_k = (0, \dots, \overset{k\text{th slot}}{\overset{\downarrow}{1}}, \dots 0)$

$$P_\pi = \begin{bmatrix} - e_{i_1} - \\ \vdots \\ - e_{i_n} - \end{bmatrix}$$

"permute the rows of identity"

$(i_1, \dots, i_n) = \pi(1, \dots, n)$

Then

$$P_\pi \cdot A = \begin{bmatrix} - e_{i_1} - \\ \vdots \\ - e_{i_n} - \end{bmatrix} \begin{bmatrix} - r_1 - \\ \vdots \\ - r_n - \end{bmatrix} = \begin{bmatrix} - r_{i_1} - \\ \vdots \\ - r_{i_n} - \end{bmatrix}$$

↑ "the one in the i_k slot of e_{i_k} grabs the i_k-row"

Thm: $P^T = P^{-1}$

$$\begin{bmatrix} - e_{i_1} - \\ \vdots \\ - e_{i_n} - \end{bmatrix} \begin{bmatrix} | & & | \\ e_{i_1} & \cdots & e_{i_n} \\ | & & | \end{bmatrix} = I \checkmark$$

<u>Thm</u>: If $A = L_1 D_1 U_1 = L_2 D_2 U_2$ then $L_1 = L_2$, $D_1 = D_2$, $U_1 = U_2$

Pf. ① $L_1 \cdot L_2 = L_3$ alway lower with 1s on diag (check)

Same for $U_1 \cdot U_2 = U_3$

② L, U alway invertibl ($\det L = \det U = 1$)

③ $$\underbrace{L_1^{-1} L_2}_{\text{lower triag 1's on diag}} \cdot D_2 = D_1 \underbrace{U_1 U_1^{-1}}_{\text{upper triag 1's on diag}}$$

$\therefore$ $L_1^{-1} L_2 = L$ must have zeros <u>below</u> diag

$$\underbrace{\begin{bmatrix} | & & | \\ c_1 & \cdots & c_n \\ | & & | \end{bmatrix}}_{L} \cdot \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix} = \begin{bmatrix} | & & | \\ d_1 c_1 & \cdots & d_n c_n \\ | & & | \end{bmatrix} = \begin{bmatrix} * & & * \\ & * & \\ 0 & & * \end{bmatrix} \checkmark$$

$\therefore\ L_1^{-1} L_2 = I \Rightarrow L_1^{-1} = L_2 = L_1 = L_2$

Same for $U_1^{-1} U_2 = id$ ✓