§1.7 Applcation: numerically compute soln's to

(*) $$-\Delta u = f \underset{1\text{-}d}{\approx} -\frac{du^2}{dx^2} = f(x) \quad 0 \le x \le 1$$

$$u(0) = u(1)$$ (2 initial condts

<u>Recall</u>: Gaussian Elimination:

$$(P)A = LDU$$

possible perm needed

$U \sim$ by Gaussian Elimination

$D \sim \text{Diag}\{d_1, \dots, d_n\}$ d_i = pivots

$L \sim I + (l_{ij})$ l_{ij} = factors

• Solve: $Ax = b \iff LDUx = b$ (Find $L, D, U \sim \frac{n^3}{3}$)

$$LDUx = b \iff DUx = L^{-1}b = c$$

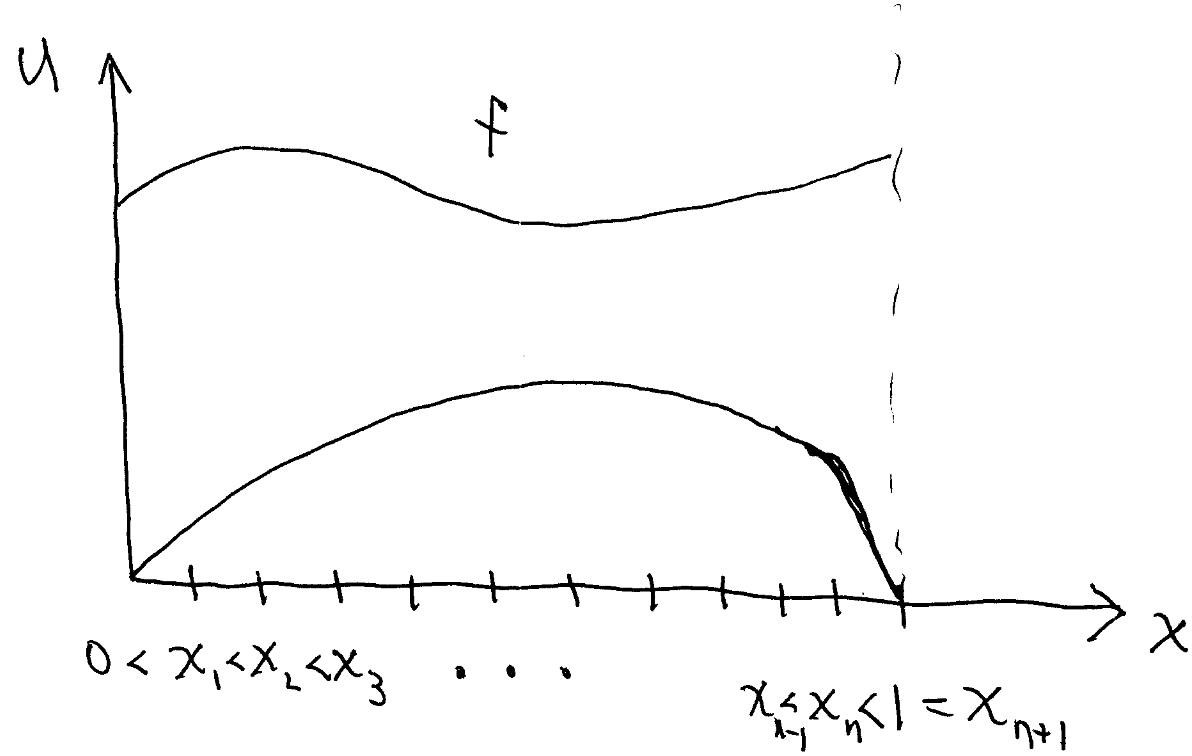
① solve $Lc = b$ for c (forward subst)

② Solve $DUx = c$ for x (back subst.)

- when $A^t = A$, $A = LUL^t$, $U = L^t$

□ Back to problem (*)

- Approx $u(x)$ by $(0, u_1, u_2, \ldots, u_n, 0) = X$



$x_j = jh$, $j = 1, \ldots, n$, $h = \frac{1}{n+1}$

$u_0 = 0 = u_{n+1}$

$f_j = f(x_j) = f(jh)$ known

Compute $(0, u_1, \ldots, u_n, 0)$ as an approx to soln $u(x)$ of (*)

- Discrete form of $-\dfrac{d^2u}{dx^2} = f(x)$:

$$\frac{\Delta u}{\Delta x} = \frac{u(x+h)-u(x-h)}{2h},\ \frac{u(x+h)-u(x)}{h},\ \frac{u(x)-u(x-h)}{h}$$

$$\frac{\Delta^2 u}{\Delta x^2} = \left[\frac{u(x+h)-u(x)}{h} - \frac{u(x)-u(x-h)}{h}\right]\frac{1}{h}$$

$$= \frac{u(x+h)-2u(x)+u(x-h)}{h^2}$$

- For our approximation this is:

$$-\frac{\Delta^2 u}{\Delta x^2} = f(x_j) \iff -\frac{u_{j+1}-2u_j+u_{j-1}}{h^2} = f(x_j)$$

$$-u_{j+1}+2u_j-u_{j-1} = f_j \quad j=1,\dots,n$$

(Takes care of $j=0, n+1$, $u_0 = 0 = u_{n+1}$)

$$-u_{j+1} + 2u_j - u_{j-1} = f_j$$

$u_0 = 0$

$$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & \cdots & 0 \\ & & & -1 & 2 & -1 & & \\ & & & & -1 & 2 & -1 & \\ & 0 & & & & \ddots & \ddots & \\ & & & & & & & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ \vdots \\ \vdots \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ \vdots \\ \vdots \\ \vdots \\ f_n \end{bmatrix}$$

$u_{n+1} = 0$

$$A \qquad x = b$$

"Tri-diagonal matrix A = zeros except one above & below diagonal"

A is a <u>symmetric</u> tridiagonal matrix

(Note advantage of centered 2nd differencing!)

Take: $n = 5$

$$\begin{array}{ccccccc} 0 & u_1 & u_2 & u_3 & u_4 & u_5 & 0 \\ \hline 0 = x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 = 1 \end{array}$$

$$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{bmatrix}$$

Note $A^t = A$

Gaussian Elimination:

$d_1 = 2$, $l_{21} = \frac{-1}{2}$
$$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ \cancel{-1} & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix} \xrightarrow{l_{32} = \frac{-2}{3}} \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ 0 & 3/2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix} \rightarrow$$

$d_2 = \frac{3}{2}$

$d_3 = 4/3$, $l_{43} = \frac{-3}{4}$
$$\rightarrow \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ 0 & 3/2 & -1 & 0 & 0 \\ 0 & 0 & 4/3 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ 0 & 3/2 & -1 & 0 & 0 \\ 0 & 0 & 4/3 & -1 & 0 \\ 0 & 0 & 0 & 5/4 & -1 \\ 0 & 0 & 0 & 0 & 6/5 \end{bmatrix}$$

$d_4 = \frac{5}{4}$, $l_{54} = \frac{-4}{5}$

In general: $l_{k+1,k} = -\dfrac{k}{k+1} \quad d_k = \dfrac{k+1}{k}$

So $\quad A = LU^*, \quad L = I + (l_{ij})$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 & 0 \\ 0 & 0 & -\frac{3}{4} & 0 & 0 \\ 0 & 0 & 0 & & 1 \end{bmatrix}$$

$$U^* = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 & 0 \\ 0 & 0 & 4/3 & -1 & 0 \\ 0 & 0 & 0 & 5/4 & -1 \\ 0 & 0 & 0 & 0 & 6/5 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & 0 & 0 \\ 0 & 0 & 4/3 & 0 & 0 \\ 0 & 0 & 0 & 5/4 & 0 \\ 0 & 0 & 0 & 0 & 6/5 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & 0 & 0 \\ 0 & 0 & 1 & -\frac{3}{4} & 0 \\ 0 & 0 & 0 & 1 & -4/5 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{U}$$

<u>Conclude</u>: $U = L^t$ $A = LUL^t$

- Much faster than $\frac{n^3}{3}$ to compute $A = LDL^T$:

Each column requires at most 2 operations:

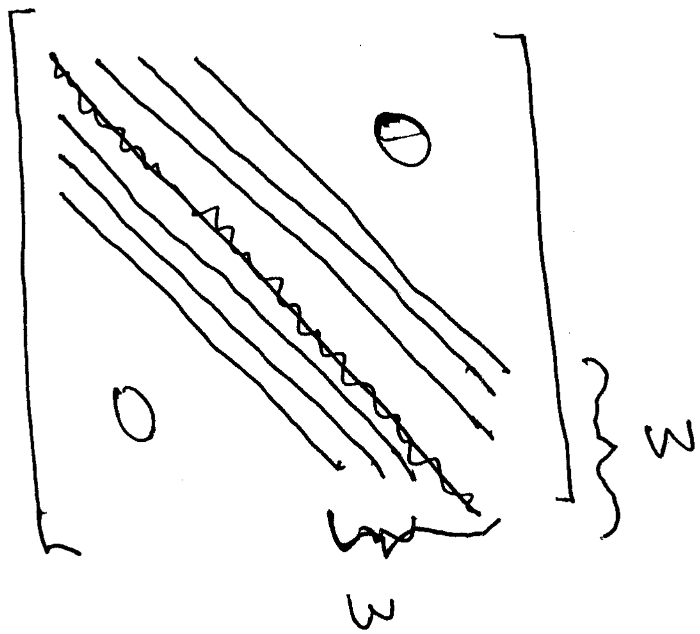
1. compute $l_{k+1,k}$
2. multiply $a_{k,k+1}$ by $l_{k+1,k}$ & subtract from $a_{k+1,k+1}$

Conclude: fewer than $2 \times n = 2n$ operations to solve $A = LUL^t$.

$$\boxed{2n \ll \frac{n^3}{3}}$$

- operation count for w-banded matrix

A =



1-banded is diagonal

2-banded is tri-diagonal

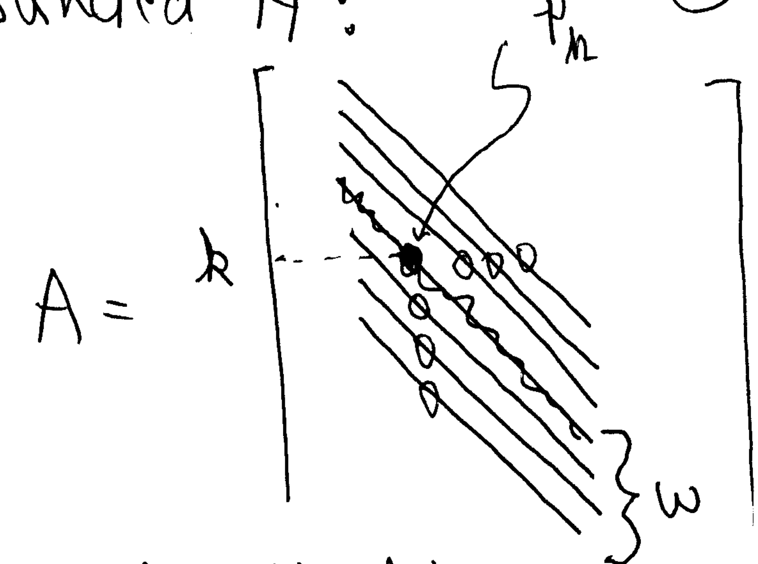
w-banded has $(w-1)$ nonzero rows below each diagonal $\sim$ $a_{ij} = 0$ if $|i-j| \geq w$

- Operation count w-banded A:

For each row:

① Compute $l_{k,k+1}$

$= 1$ operation



② "mult $a_{k,l}$ by $l_{k,k+1}$ and subtract" $1 \le l \le w$

$= w-1$ operations

w-operations to reduce each row under P_k with nonzero leading entry

$\Rightarrow$ $(w)\cdot(w-1)$ operations for each P_k

# of op's

nonzero rows below P_k

$\Rightarrow$ $w(w-1)\cdot n$ total operations

$\approx$ $w^2 n$ operation to row reduce w-banded

<u>Comments</u>: $\Delta u = f$ not so simple for dim > 1. Many ideas in research. ~ "Multigrid Methods"