§2.1

Overview of Vector Spaces for Linear Algebra--Temple

(6) (1)

Vector Spaces –

- A vector space is a ~~space~~ set "closed under addition & scalar mult" $\Leftrightarrow$

V is "closed under taking linear comb's"

$$\{V_1, \ldots, V_n\} \subseteq V \Rightarrow \sum_{i=1}^{n} c_i V_i \in V$$

- 2 kinds: ∞-dimensional { real #, functions }

finite-dimensional < Matrices, $\mathbb{R}^n$ vectors

Ex: $\mathbb{R}^n = \{(x_1, \ldots, x_n) : x_k \in \mathbb{R}\}$ is a vector space

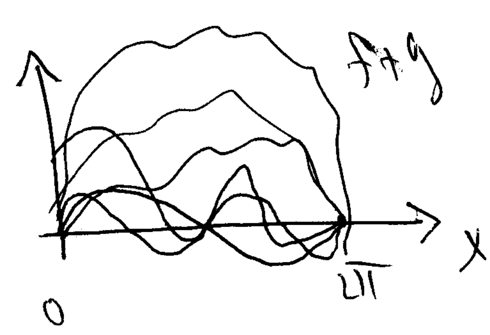
$-5x = (-5x_1, \ldots, -5x_n)$ "Dimension = n"

$x + y = (x_1+y_1, \ldots, x_n+y_n)$

Ex: 2π-periodic continuous functions

Eg $f(x) = \sin x, \sin 2x, \sin 3x \cdots$

∞-dimensional



- We only care about finite-d vector space

Abstract Defn: V vector space if + & scal mult are defined & satisfy

1. + is commutative & assoc $u+v=v+u$ $(u+v)+w=u+(v+w)$
2. $\cdot$ is comm & assoc $(2\cdot3)u=u\cdot 23=2\cdot(3u)$
3. $\cdot$ distributes over addition $2(u+v)=2u+2v$ etc
4. $\exists$ additive inverses $-u$ & additive identity 0
5. $1\cdot v=v$
6. Closed under these operations

Eg Thm① $(-1)v=-v$ $\quad ((-1)\cdot v+1\cdot v)=(-1+1)\cdot v$
$=0\cdot v=0$ (by)

Thm② $0\cdot v=0$

Pf $0\cdot v=(0+0)\cdot v=0\cdot v+0\cdot v$

$0\cdot v-0\cdot v=0\cdot v+0\cdot v-0\cdot v$

$0=0\cdot v$ ✓

④ Basic Defns

Defn ① V a vector space. We say $\{v_1, \dots v_n\}$ spans V if every element $v \in V$ is a linear comb. of $v_1 \cdots v_n$:

$$\exists\ c_k \text{ st } \quad v = \sum_{k=1}^{n} c_n v_n$$

Eg. $(1,0)$ & $(1,1)$ span $\mathbb{R}^2$

$$c_1(1,0) + c_2(1,1) = (x,y)$$

$$\begin{aligned} c_1 + c_2 &= x \\ 0 + c_2 &= y \end{aligned} \qquad \boxed{\begin{aligned} c_1 &= x - y \\ c_2 &= y \end{aligned}}$$

"Finding linear combs $\Leftrightarrow$ solving linear eqn's"

Defn ② $\{v_1, \dots, v_n\}$ are linearly dependent if $\exists\ c_1, \dots, c_n$ st $\sum_{k=1}^{n} c_n v_n = 0$ not all $c_k = 0$

Defn 3 $\{v_1, \ldots, v_n\}$ are linearly independent if they are NOT linearly dependent

I.e. if not all $c_k = 0$, then $\sum_{k=1}^{n} c_k v_k \neq 0$

Defn 4 A linearly independent spanning set is called a basis

Defn 5 V is said to be finite dimensional ~~if it has a finite basis~~ ~~is spanned by a finite # of elements~~

if $\exists$ finite set $\{v_1, \ldots, v_n\} \subseteq V$ st $\{v_1, \ldots, v_n\}$ spans V.

Basic Theorems:

Thm 1 If V is finite dimensional, then it has a basis, and every basis has the same # of ele's. This # is called the *dimension* of $V \equiv \dim(V)$

Thm 2 Every Finite Dimensional Vector space is *isomorphic* to $\mathbb{R}^n$, $n = \dim(V)$

~~E.g. $\{v_1, \ldots, v_n\}$ basis for V,~~

~~map $\Phi : v_i \mapsto e_i$ $\Rightarrow$~~

$$\cancel{\Phi\left(\sum_{k=1}^{n} c_k v_k\right) = \sum_{k=1}^{n} c_k \Phi(v_k)}$$

I.e. Given basis $\{v_1, \ldots, v_n\} \subseteq V$,

$\exists!$ way to write $v = \sum_{k=1}^{n} c_k v_k$

scalars in $\mathbb{R}$

If you map $v \overset{\Phi}{\longleftrightarrow} (c_1, \ldots, c_n)$

$\Phi : V \longrightarrow \mathbb{R}^n$

"The mult & addition in V is represented in the mult & addition of the components"

- We study matrices & the vector spaces associated with the rows & colms & soln's $Ax = b$

Thm 3 $\dim C(A) = \dim C(A^t)$

$C(A)$ = Vector space spanned by colms A

$C(A^t)$ = Vector space spanned by rows A

Thm 4 If $A_{m \times n}$, then

$$\dim C(A) + \dim K(A) = n$$

$K(A) \equiv$ Kernel of A = Vector space

$$= \{ x \in \mathbb{R}^n : \underset{(m \times n)(n \times 1)}{Ax} = 0 \}$$

To Demonstrate These we use Gaussian Elimination & Row Echelon Form