§2.1–2.4

The theory of vector spaces, and the fundamental theorems on Row(A), Col(A), Ker(A)---Temple

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- Vector Space V: "A set with + & scalar · defined satisfying ① comm, ② assoc, ③ dist, ④ 0, ⑤ inverse & ⑥ $1 \cdot v = v$

- Lemma ① Any subset of V closed under linear combinations is a vector space (subspace).

Pf. +, · defined ∀ cv, $v+w$ & ①–⑥ hold ✓

- Lemma ②: Given $\{v_1, \dots, v_n\} \subseteq V$,

$$\mathrm{Span}\{v_1, \dots, v_n\} = \left\{ \sum_{k=1}^{n} c_k v_k : c_k \in \mathbb{R} \right\}$$

is a subspace of V (the smallest subspace containing $v_1, \dots, v_n$)

Pf. "closed under + & ·" ✓

- Defn: A linearly indept set $\{v_1, \dots, v_n\}$ that spans V is called a basis

Thm 1: If V is finite dimensional (has a finite spanning set), then it has a basis, and every basis has same # of elements (dimension of V)

Thm 2: $\exists!$ to express $v \in V$ in terms of a basis: $v = \sum_{i=1}^{n} c_i v_i = \sum_{i=1}^{n} d_i v_i$ for basis $\{v_1, \dots, v_n\} \Rightarrow c_i = d_i$.

- Let $A_{m \times n} = \begin{bmatrix} - r_1 - \\ \vdots \\ - r_m - \end{bmatrix} = \begin{bmatrix} | & & | \\ c_1 & \cdots & c_n \\ | & & | \end{bmatrix}$

(Eg $A = \begin{bmatrix} 2 & 3 & 1 \\ -1 & 0 & 1 \end{bmatrix}$ $\begin{matrix} r_1 = (2,3,1) \\ r_2 = (-1,0,1) \end{matrix}$; $c_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ $c_2 = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$ $c_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$)

$\mathrm{Span}\{c_1, \dots, c_n\}$ = "Column Space" $\equiv C(A) =$ Rang(A)

$\mathrm{Span}\{r_1, \dots, r_m\}$ = "row space" $= C(A^t)$

Thm 3 $\quad \dim C(A) = \dim C(A^t) =$ rank of the matrix

"dim colm space = dim row space"

Thm 4 $\quad \dim C(A) + \dim K(A) = n$ $\nwarrow$ dim of $\mathbb{R}^n$ where $K(A) \subseteq \mathbb{R}^n$

$C(A) \subseteq \mathbb{R}^n \Rightarrow \dim K(A) = n - r$ $\nwarrow$ rank of matrix

Here: $K(A) = \{x : Ax = 0\} \equiv$ Kernel of A

Note: $A(c_1 x_1 + c_2 x_2) = c_1 A x_1 + c_2 A x_2$

so if $Ax_1 = 0$ & $Ax_2 = 0$ then linear comb's of x_1 & x_2 satis $A(c_1 x_1 + c_2 x_2) = 0 \Rightarrow$

$K(A)$ closed under $+$ & $\cdot$ $\Rightarrow$ vector space

subspace of $\mathbb{R}^m$

(E) Theorems (1) - (4) are proven via Gaussian Elimination & Row echelon Form of A:

- Assume $A_{m \times n}$: For specific example

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}_{3 \times 4}$$

- Even Tho A not square, we can apply Gaussian Eliminatn same way:

$i \neq j$; $E_{ij}(a)$ = "identity matrix with a in ij'th entry"

= $(m \times m)$, square, m = # rows of A

Then: $E_{ij}(a)A$ = "the matrix obtained from A by adding a times row j to row i"

Pf. Same as before - just check

Eg
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & 0 & 1 \end{bmatrix} \begin{bmatrix} - & r_1 & - \\ - & r_2 & - \\ - & r_3 & - \end{bmatrix} = \begin{bmatrix} - & r_1 & - \\ - & r_2 & - \\ - & ar_1 + r_3 & - \end{bmatrix}$$

Conclude: by the same procedure as for square matrices, we can multiply by $E_{ij}(-l_{ij})$ to make zeros under all the pivots.

Along the way we may have to interchange rows (Mult by permutation matrix P_{ij}) and we may get zero pivots ...

Eg:

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ -1 & -3 & 3 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & 6 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ pivots 1, 3

(L1) By multiply thru by a sequence of $E_{ij}(a)$'s & P_{ij}'s we can take any matrix $A_{m \times n}$ to <u>eschelon</u> <u>fm</u>

$$\begin{bmatrix} p_1 & * & * & * & * & * & * & * \\ 0 & p_2 & * & * & * & * & * & * \\ 0 & 0 & 0 & p_3 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- pivot is 1st nonzero entry in each row
- below pivots are zeros
- each pivot lies to rt of pivot in previous row

(L2) By mult thru by a diagonal matrix we can make all pivots $= 1$

Eg: $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Eg: in general: $$D = \begin{bmatrix} 1/p_1 & & & & & \\ & 1/p_2 & & & 0 & \\ & & \ddots & & & \\ & & & 1/p_k & & \\ & 0 & & & 1, & \\ & & & & & 1 \end{bmatrix}_{m \times m}$$

D invertible with inverse $\mathrm{diag}(p_1, p_2, \ldots, p_k, 1, \ldots, 1)$

~~(L3) By mult by ele matrices $E_{ij}(a)$ $j > i$ we can make zeros above every pivot.~~

(L3) By mult A on left by elementary matrices $E_{ij}(a)$, $j>i$, we can make zeros above evey pivot:

Eg
$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$E_{12}(-3)$ = "subtract 3 x row-2 from row-1

<u>Thm</u>: By mult A on the left by a sequene of elementay matrices $E_{ij}(a)$, P_{ij}, D, we can take A to <u>reduced</u> <u>row eschelon form</u>:

- pivot = 1st nonzero enty in each row = 1
- above & below pivots = 0
- each pivot lies to rt of pivot in previous row

$$\begin{bmatrix} 1 & * & 0 & * & * & 0 & * & 0 \\ 0 & 0 & 1 & * & * & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

That is $\forall\ A_{m\times n}$ we have

$$E_1 \cdot E_2 \cdots E_N A = R$$

where R is in RRF & E_i are elementary matrices

Now here is the point:

Theorem (A) Multiplication of A by an $m \times m$ elementary matrix on left does NOT

① Change the row space $C(A^T)$

② Change the solution space $= K(A)$

~~(B) Change any subset of columns from linearly dependent to linearly indep't or vise versa.~~

"<u>Proof</u>" Adding a mult of row j to row i does not chng row space or soln space

i.e. $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & 0 & 1 \end{bmatrix} \begin{bmatrix} - r_1 - \\ - r_2 - \\ - r_3 - \end{bmatrix} = \begin{bmatrix} - r_1 - \\ - r_2 - \\ - ar_1 + r_3 - \end{bmatrix}$$

$$E_{31}(a) \cdot A = E_{31}A$$

$$\mathrm{Span}\{r_1, r_2, r_3\} = \mathrm{Span}\{r_1, r_2, ar_1 + r_3\}$$

$$r_3 = -ar_1 + (ar_1 + r_3) \checkmark$$

$$(ar_1 + r_3) = ar_1 + r_3 \checkmark$$

Alo $$\begin{bmatrix} - r_1 - \\ - r_2 - \\ - r_3 - \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = 0 \text{ iff } \begin{bmatrix} - r_1 - \\ - r_2 - \\ - ar_1 + r_3 - \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = 0$$

$\Rightarrow$ Row Space & Soln Space unchanged. Similarl for P_{ij} & D

Theorem (B): The nonzero rows of R give a basis for the row space of R

Pf. Since the other rows are zero, the non zero rows span the row space ✓

Now assume

$$R = \begin{bmatrix} - & r_1 & - \\ & \vdots & \\ - & r_r & - \\ - & 0 & - \\ - & 0 & - \end{bmatrix}.$$

If $\sum_{i=1}^{r} c_i r_i = 0$, then $c_i = 0$

because only r_i has a nonzero entry in the colm of the ith pivot! ✓

Eg
$$\begin{array}{c} c_1 x \\ c_2 x \\ c_3 x \\ c_4 x \\ \end{array} \begin{bmatrix} 1&0&*&*&0&*&0\\ 0&1&*&*&0&*&0 \\ 0&0&0&0&1&*&0\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0\end{bmatrix} \begin{array}{l} \leftarrow r_1 \\ \leftarrow r_2\\ \leftarrow r_3 \\ \leftarrow r_4 \\ \leftarrow 0\end{array} \qquad (R)$$

$$\begin{array}{ccccccc} \uparrow & \uparrow & & & \uparrow & & \uparrow \\ c_1=0 & c_2=0 & & & c_3=0 & & c_4=0 \end{array}$$

<u>Conclude</u>: $\dim C(A^t)$ = # of nonzero rows in R = # pivots = r

From R we can find the solution space $Ax=0$ iff $Rx=0$.

<u>thm</u>(c): The $K(R)$ has a basis with $n-r$ elements = # of non-pivot = free var's

"pf"

<u>Example</u>: $A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix} \to R = \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$Rx = 0 \iff \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = 0$$

$$\iff \begin{aligned} u + 3v + 0w - 1y &= 0 \\ 0u + 0v + w + y &= 0 \end{aligned}$$

u = 1-pivot variable w = 2-pivot variable

Since pivots are $=1$ & have 0's above & below them, you can always solve the ith non zero equation for the ith "pivot variable" in terms of the "non-pivot variables" = "free variables"

$$\Leftrightarrow \quad u = -3v + 1y$$

$$w = -y$$

<u>Soln</u>: $$\begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} -3v + y \\ v \\ -y \\ y \end{bmatrix} = v \overset{\vec{V}_2}{\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}} + y \overset{\vec{V}_4}{\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}}$$

eliminate pivot variable

The identity matrix in the non pivot varables

In general: $K = \text{Span}\{\vec{V}_{i_1}, \dots, \vec{V}_{i_{n-r}}\}$

one vector for each non-pivot variable with id matrix in pivot variable

~~Thus $\sum_{i=1}^{n-r} c_i v_i$ has $\sum c_i$ in pivot variable $\Rightarrow = 0$ iff~~ if all $c_i = 0$.

Thus: The soln of $Ax=0$ is

$$\underbrace{x = \sum}_{\text{sum over non-pivot = free variables } x_i} x_i \underbrace{\vec{V}_i}_{\text{const vector associate with the free variable } x_i}$$

But $V_i = 1$ in ith component, and

$V_j = 0$ in ith comp for $j \neq i$

$(V_j)_i = \delta_{ij}$ when ith entry corresponds to the free variable x_i

Conclude: $\{V_i\}$ are indept & span $K(R) = K(A)$

I.e., they clearly span, and independence follows

because if $0 = \sum c_i V_i$ ← (c_i in ith comp)

(i assoc w. free variables x_i) $\Rightarrow c_i = 0$ every i

- Cor (1) of Thm (c): If A is an $m \times n$ matrix st $n > m$, then $Ax = 0$ has a non-zero soln.

Pf. Since $\exists$ $m < n$ rows, the dim r of the row spaces satis $r \leq m < n$. Thus by Thm (c) $\exists$ a basis for $K(A)$ of dim $n - r > 0$ $\Rightarrow$ $\exists$ nonzero soln's ✓

- Cor (2) of Thm (c): Any two bases for a vector space V have the same # of elements

Pf. Assume $\{v_1, \ldots, v_m\}$ & $\{w_1, \ldots, w_n\}$ are bases for V.

Then $\forall$ w $\exists$ a_{ij} st $w = \sum_{i=1}^{m} a_{ij} v_i$

Or in matrix form

$$[w_1, \dots, w_n] = [v_1, \dots, v_m] \begin{bmatrix} | & | & & | \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ | & | & & | \end{bmatrix} \quad (*)$$

Sum v_i against a_{i1} to get the w_1 linear com

But $n > m \Rightarrow \exists$ $x \neq 0$ st $Ax = 0$. Mult (*) on right by $x \Rightarrow$

$$[w_1, \dots, w_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = [v_1, \dots, v_m] A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = 0$$

$\Rightarrow \sum_{i=1}^{n} x_i w_i = 0 \Rightarrow w_i$ <u>not</u> lin indept $\times\!\!\!\times$ $\therefore m = n$ ✓

• Cor (3): If r = dim row space of A = dim $C(A^T)$, then

$$\dim K(A) = n - r$$

Pf. By Thm C, $K(A)$ has a basis of length $n-r$, $\therefore \dim K(A) = n-r$ ✓

~~• Cor (4): dim $C(A)$ = r = dim of row space~~

~~Pf.~~

• In order to prove Thm 3, we need one more Corollary:

Cor (4): If $A = E_1 \cdots E_N R$ where $R = \mathrm{rref}(A)$, then any subset of the columns of A are linearly indept iff the corresponding colns of R are linearly indept.

Pf. Let $\underset{m\times n}{A} = \begin{bmatrix} | & & | \\ c_1 & \cdots & c_n \\ | & & | \end{bmatrix}$ $\quad R = \begin{bmatrix} | & & | \\ \bar{c}_1 & \cdots & \bar{c}_n \\ | & & | \end{bmatrix}$.

By Thm(A) we have

$$Ax = 0 \text{ iff } Rx = 0.$$

Thus, $\{c_{i_1}, \ldots, c_{i_k}\}$ are linearly dept

iff $\sum_{i=1}^{n} x_i c_i = 0$ where, not all $c_{i_k} = 0$, but $x_i = 0$ for $i \notin \{i_1, \ldots, i_k\}$.

iff $Ax = 0$ iff $Rx = 0$ iff $\sum_{i=1} x_i \bar{c}_i = 0$

iff $\{\bar{c}_{i_1}, \ldots, \bar{c}_{i_k}\}$ are linearly dept ✓

Finally we can prove:

Thm 3 : $\dim C(A) = r = \dim C(A^T)$

Pf. By the structure of R given in (R), the colms of R with the pivots form a basis for the column space $C(R)$ of R.

By Cor(4), the corresponding colms of A ~~are~~ form a basis of $C(A)$; i.e, they are linearly indept, and span the space because any larger set of colms is linearly dependent ✓

That is the cleanest route to the Fund Thms of Matrix Algebra . . .

Further Corollaries:

<u>Cor</u> (5): Every set of linearly indept vectors can be completed to a basis

<u>Cor</u> (6): Every set of vectors that spans V contains a basis for V

<u>Cor</u>(6): If $\{v_1, \ldots, v_n\}$ are a basis for V,

and $w = c_1 v_1 + \cdots + c_n v_n = \bar{c}_1 v_1 + \cdots + \bar{c}_n v_n$

then $c_i = \bar{c}_i$. ($\exists!$ way to express an ele of V in terms of basis.

Pf. assume not, $c_i \neq \bar{c}_i$. Then

$\sum_{i=1}^{n} (c_i - \bar{c}_i) v_i = 0$ # $\{v_i\}$ lin indept