E The soln space of inhomogeneous eqn 8

$$\underset{(m\times n)}{A}\ \underset{(n\times 1)}{x} = \underset{(m\times 1)}{b}$$

Thm 1: To have any soln, b must lie in colm space of A, $b \in C(A)$

Pf.

$$\begin{bmatrix} | & & | \\ c_1 & \cdots & c_n \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} | \\ c_1 \\ | \end{bmatrix} + \cdots + x_n \begin{bmatrix} | \\ c_n \\ | \end{bmatrix} = b$$

$\Leftrightarrow$ "b is a linear comb of colms of A"

$\Leftrightarrow$ $b \in \text{Span}\{c_1, \cdots, c_n\}$ $\Leftrightarrow$ $b \in C(A)$ ✓

<u>Thm 2</u>: If $\cancel{x}_p$ is a <u>particular</u> soln of

$$Ax_p = b,$$

then every soln is given by $x = x_n + x_p$

where $x_n \in K(A) \iff Ax_n = 0$

$$K(A) \equiv \text{Kernel of } A \equiv \text{nullspace of } A = \{x : Ax = 0\}$$

<u>Recall</u>: $\dim K(A) = n - r \qquad r = \text{rank}(A)$

Pf. Assume $Ax_p = b$. Let $\bar{X} = \{x : Ax = b\} = $ Soln Space.

(1) If $x_n \in K(A)$, then $A(x_p + x_n) = \underbrace{Ax_p}_{b} + \underbrace{Ax_n}_{0} = b$ ✓

$\Rightarrow \cancel{AA}\ x_p + K(A) \subseteq \bar{X}$

(2) If $\cancel{Ax=0}$ $x \in \bar{X}$,

then $Ax = b \Rightarrow A(x - x_p) = Ax - Ax_p = b - b = 0$

$\therefore x_n = x - x_p \in K(A)$ and $x = x_n + x_p \Rightarrow \bar{X} \subseteq x_p + K(A)$

$\therefore \bar{X} = x_p + K(A)$ ✓

- How to find x_p : $A = \begin{bmatrix} | & & | \\ c_1 & \cdots & c_n \\ | & & | \end{bmatrix}$

- Construct augmented matrix

$$[A ; b] = \begin{bmatrix} | & & | & | \\ c_1 & \cdots & c_n & b \\ | & & | & | \end{bmatrix}_{m \times (n+1)}$$

- Let $E_1 \cdots E_N$ be the elementary matrices that take A to $\mathrm{rref}(A) = R$:

$$\underbrace{E_1 \cdots E_N}_{E} A = R$$

- Then by how colms of A act on matrix to left

$$E_1 \cdots E_N [A ; b] = \underbrace{E_1 \cdots E_N}_{E} \begin{bmatrix} | & & | & | \\ c_1 & \cdots & c_n & b \\ | & & | & | \end{bmatrix}$$

$$= \begin{bmatrix} | & | & & | & | \\ Ec_1 & Ec_2 & \cdots & Ec_n & Eb \\ | & | & & | & | \end{bmatrix} = [R ; Eb]$$

- But $Ax = b \Leftrightarrow EAx = Eb \Leftrightarrow Rx = Eb$

$\therefore$ to find x_p we need only solve

$$Rx_p = Eb$$

- To see the simplest soln:

$$\begin{bmatrix} 1 & 0 & * & * & 0 & * & 0 \\ 0 & 1 & * & * & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 1 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \\ \bar{b}_3 \\ \bar{b}_4 \\ \bar{b}_5 \end{bmatrix} = Eb$$

Set the non-pivot = free variables $= 0 = x_3 = x_4 = x_6$

Set the pivot variables $x_i = \bar{b}_i$

<u>Soln</u>: $x_p = \begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \\ 0 \\ 0 \\ \bar{b}_3 \\ 0 \\ \bar{b}_7 \end{bmatrix}$

- Q: What is the "solvabity condition" for b?

<u>Ans</u>: Since in the <u>pivot variable</u> R is id, we can find x s.t. $Rx = b$ <u>in the nonzero rows</u>. However, to get equaty in the zero rows, we need

$$(Eb)_i \equiv \bar{b}_i = 0 \qquad i = r+1, r+2, \ldots, m.$$

In general: if $E = \begin{bmatrix} - \vec{r}_1 - \\ \vdots \\ - \vec{r}_m - \end{bmatrix}_{m \times m}$ then

$$\bar{b} = E \cdot b = \begin{bmatrix} \vec{r}_1 \cdot b \\ \vdots \\ \vec{r}_m \cdot b \end{bmatrix}$$

So solvabity condt is

$$\vec{r}_{\underbrace{r}_{\text{rank of } A}+1} \cdot b = 0, \ldots, \vec{r}_m \cdot b = 0$$

- <u>Note</u>: you can get solvabity condt directly from

$$L^{-1}A = U \overset{eg}{=} \begin{bmatrix} -3 & 3 & 1 & -1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = L^{-1}b$$

$$Ax=b \iff Ux = L^{-1}b = c$$

solvable iff $(L^{-1}b)_{r+1} = \cdots = (L^{-1}b)_m = 0$

$$c_{r+1} = \cdots = c_m = 0 \quad \checkmark$$

<u>Summary</u>: "The zero rows of U or R give the solvabity condt on b"

<u>Ex</u>: Find the solvabity condt on b st

$$Ax = b$$

has a soln if

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$$

From before:

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix} \xrightarrow{E_{21}(-2)} \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ -1 & -3 & 3 & 4 \end{bmatrix} \xrightarrow{E_{31}(1)} \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & 6 \end{bmatrix}$$

$$\xrightarrow{E_{32}(-2)} \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U \qquad \text{rank} = 2 = r$$

$\therefore Ax = b \iff$

$$E_{32}(-2)\,E_{31}(1)\,E_{21}(-2)\,A\,x = Ux = E_{32}(-2)\,E_{31}(1)\,E_{21}(-2)\,b$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -2 & 1 \end{bmatrix} b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} b$$

$\Rightarrow$ Solvabity condt

$$\boxed{5b_1 - 2b_2 + b_3 = 0}$$

Eg if $\exists$ 3 zero rows $\Rightarrow$ 3 solu condts etc.

Matrices of rank-1

Ex

$$\underset{(1\times 2)}{[1, 2]} \underset{(2\times 1)}{\begin{bmatrix} -1 \\ 3 \end{bmatrix}} = \underset{(1\times 1)}{-1+6 = 5}$$

$$\underset{(2\times 1)}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} \underset{(1\times 2)}{[-1, 3]} = \begin{bmatrix} -1 & 3 \\ -2 & 6 \end{bmatrix} = \text{"rank-1 matrix"}$$

In general: given $\begin{bmatrix} | \\ \vec{c} \\ | \end{bmatrix}$ & $[-\vec{r}-]$

$$\underset{(m\times 1)}{\begin{bmatrix} | \\ \vec{c} \\ | \end{bmatrix}} \underset{(1\times n)}{[-\vec{r}-]} = \underset{m\times n}{\begin{bmatrix} | & | & & | \\ r_1\vec{c} & r_2\vec{c} & \cdots & r_n\vec{c} \\ | & | & & | \end{bmatrix}} = \begin{bmatrix} - & c_1\vec{r}_1 & - \\ - & c_2\vec{r}_1 & - \\ & \vdots & \\ - & c_m\vec{r}_m & - \end{bmatrix}$$

Defn: given vector $\vec{c} \in \mathbb{R}^m$ & $\vec{r} \in \mathbb{R}^n$

the $m \times n$ ~~[illegible]~~ matrix $\vec{c} \cdot \vec{r}^T$

is called a rank-1 matrix

- "colm's are mult's of $\vec{c}$"
- "rows are mult's of $\vec{r}$"

Preview: SVD $\quad A = U \Sigma V^T = \sum_{i=1}^{n} \sigma_i \vec{u}_i \vec{v}_j^T$

ON. rank-1 matries

Inverse Matrices

Defn: B is a left inverse of A if $BA = I$

B is a right inverse of A if $AB = I$

Defn: we say A has an inverse A^{-1} if

$A^{-1}A = I = AA^{-1}$ "A^{-1} is both left/right"

Thm: If A has a left & right inverse, then they are equal

Pf. $BA = I$ & $AC = I$ $\Rightarrow$ $B(AC) = B$

$(BA)C = B \Rightarrow C = B$ ✓

Thm: If A is $n \times n$, then A^{-1} exists iff

① $|A| \neq 0$

② $\exists$ n nonzero pivots

- What about $A_{m \times n}$ $m \neq n$

"can only have rt or left not both"

"for rt or left to exist, rank(A) = min(m,n)"

Ex

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

(2×3) (3×1) (2×1)

Look for B st $B_{3\times 2} \cdot A_{2\times 3} = id_{3\times 3} \Rightarrow$
can solve $Ax = b$ by ...

$$\underbrace{B\, A}_{Id_{3\times 3}}\, \underset{\sim}{x} = B\, \underset{\sim}{b}$$

$(3\times 2)(2\times 3)(3\times 1)$ (2×1)

But B cannot exist:

$$\begin{bmatrix} - b_1 - \\ - b_2 - \\ - b_3 - \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} - e_1 - \\ - e_2 - \\ - e_3 - \end{bmatrix}$$

" $e_1 \in$ span $(1,2,-1)$ & $(2,1,1)$

$e_2 \in$ Span " "

$e_3 \in$ " " "

\# rows of A only span 2-d space.

Thm $\Rightarrow$ if $A_{m \times n}$, $m < n \Rightarrow$ no left inv.

if $A_{m \times n}$, $m > n \Rightarrow$ no right inv.

<u>Thm</u>: If $A_{m\times n}$, $m>n$, then A has a left inverse iff A has max'al rank n

If $A_{m\times n}$, $m<n$ then A has a rt inv iff A has max'al rank n

$$\begin{bmatrix} - r_1 - \\ - r_2 - \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

2×3 (3×2) (2×1)

Clearly we can solve

$$\begin{bmatrix} - r_1 - \\ - r_2 - \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

if rows of A span a 2-d space $\Rightarrow$ rank$(A)=n=2$

" $r_1=(a_1,b_1,c_1) \Rightarrow$ $a_1(1,2) + b_1(2,1) + c_1(-1,1) = (1,0)$ || $a_2(1,2) + b_2(2,1) + c_2(-1,1) = (0,1)$ ✓