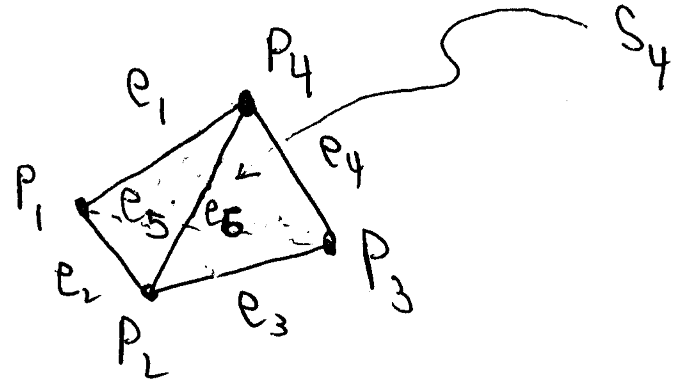
§2.5 Euler Formula

- Consider a "polytope" = multisided convex polygon

Eg:



vertices: P_1, P_2, P_3, P_4

edges: $e_1, e_2, e_3, e_4, e_5, e_6$

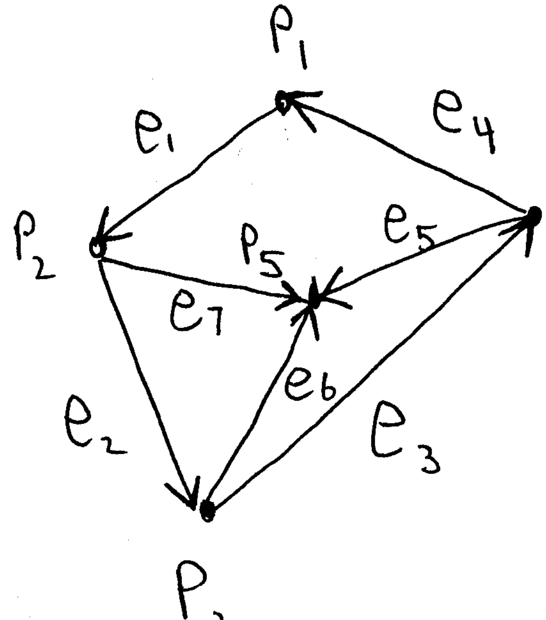
sides: S_1, S_2, S_3, S_4

Euler Formula: # vertices − # edges + # sides = 2

Eg: $4 - 6 + 4 = 2$ ✓

<u>Thm</u>: Eulers formula is a consequent of row rank = col rank!

- More generally – consider any connected graph —



~ "projection of polytome onto 2-d plane"

We construct the <u>incidence matrix</u> (edge-node) of graph

① name the nodes/vertices $P_1 P_2 \cdots P_n$

② give a vector direction to edges

Assign #s to the vertices & vectors to the edges eg $P_1 \Leftrightarrow 1, \ldots, P_k \longleftrightarrow k$, $e_1 \longleftrightarrow \vec{V}_1, 2 \ldots, e_k \longleftrightarrow \vec{V}_k$

$\vec{V}_1 = (-1, 1, 0, 0, 0)$ "e_1 points from P_1 to P_2"

$\vec{V}_7 = (0, -1, 0, 0, 1)$ "e_7 pts from P_2 to P_5"

etc

For example: "edge not incidence matrix"

$$
\begin{array}{c}
\vec{v}_1 \\ \vec{v}_2 \\ \vec{v}_3 \\ \vec{v}_4 \\ \vec{v}_5 \\ \vec{v}_6 \\ \vec{v}_7 \\ \\
\end{array}
\begin{array}{c}
\begin{bmatrix}
-1 & 1 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 \\
1 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 & 1 \\
0 & 0 & -1 & 0 & 1 \\
0 & -1 & 0 & 0 & 1
\end{bmatrix} \\
\begin{array}{ccccc} P_1 & P_2 & P_3 & P_4 & P_5 \end{array}
\end{array}
= \underset{7\times 5}{A} = \underset{m\times n}{A}
$$

Theorem: If the graph is connected, then the rank $\underset{m\times n}{A} = n-1$

Pf. 1st, sum of colms $= 0$ (because one -1 & one $+1$ in each row). Thus Rank $A \le n-1$. (Colms are linearly indept.) I.e.

$$\underset{7\times 5}{A} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 0$$

Claim: if graph is connected, then $\operatorname{rank}(A) = n-1$.

Defn: graph is connected if any two nodes can be reached by a connected sequence of edges. So assume

$$\underset{7\times 5}{A}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = 0$$

Row 1 $\Rightarrow$ $x_1 = x_2$

Row 2 $\Rightarrow$ $x_2 = x_3$

Row 3 $\Rightarrow$ $x_3 = x_4$

Row 4 $\Rightarrow$ $x_1 = x_4$

$\vdots$

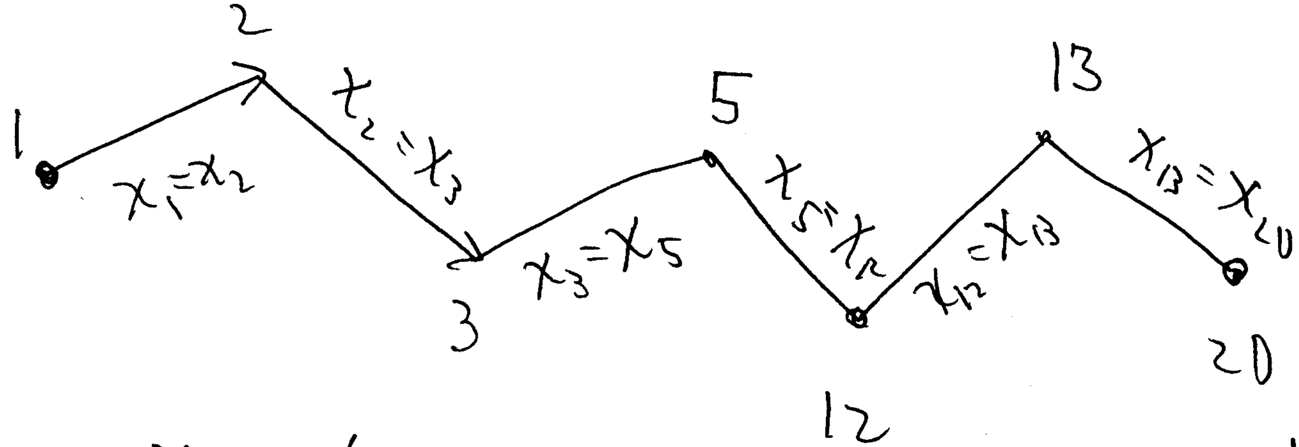
Row k $\Rightarrow$ "the two #'s corresponding to endpts of edge k are equal"

As long as all nodes can be connected by a sequence of edges we get

$x_1 = x_2 = \cdots = x_m = c \Rightarrow x = c\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \Rightarrow$ 1-d Kernel

4B

Ie



$\Rightarrow x_1 = x_{20}$ ($x_1 = x_k \ \forall k$ by same argument)

$$0 = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{array}{c} \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ x_4 - x_3 \\ x_1 - x_4 \\ x_5 - x_3 \\ x_5 - x_2 \end{bmatrix} \end{array} \begin{array}{l} \leftarrow x_1 = x_2 \\ \leftarrow x_3 = x_2 \\ \leftarrow x_4 = x_3 \\ \leftarrow x_4 = x_1 \\ \leftarrow x_5 = x_3 \\ \leftarrow x_5 = x_2 \end{array}$$

<u>Conclude</u>: $1 = \dim \operatorname{Ker}(A) = n - r$

$$\boxed{r = n-1} \checkmark$$

- Application to polytopes –

$A_{m\times n} \Rightarrow$ m-edges & n-vertices

Q: what about sides?

Ans: sides correspond sums of $\vec{v}$'s that sum to zero

<u>Eg</u>: Side $\overline{P_2P_3P_5} \Leftrightarrow \vec{V}_2 + \vec{V}_6 - \vec{V}_7 = 0$ (a)

Side $\overline{P_1P_2P_5P_4} \Leftrightarrow \vec{V}_1 + \vec{V}_7 + \vec{V}_5 + \vec{V}_4 = 0$

Side $\overline{P_3P_4P_5} \Leftrightarrow \vec{V}_3 + \vec{V}_5 - \vec{V}_6 = 0$ (b)

Some of these are redundant: ie

(a)+(b) $\Leftrightarrow \vec{V}_2 + \vec{V}_3 + \vec{V}_5 - \vec{V}_7 = 0$

<u>Defn</u>: The number of sides = # of independent sums of $\vec{v}$'s that give zero, ie. the dimension of the kernel of A^t:

$$[x_1, \dots, x_7] \begin{bmatrix} - & \vec{v}_1 & - \\ & \vdots & \\ - & \vec{v}_7 & - \end{bmatrix} = 0$$

$$\Leftrightarrow \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_7 \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_7 \end{bmatrix} = 0$$

$$\dim \operatorname{Ker}(A^t) = \# \text{ sides} = m - r = m - (n-1)$$

$$= \underset{\#\text{edges}}{m} - \underset{\#\text{vertices}}{n} + 1$$

one side left unaccounted for - the <u>back</u> side

$$\#\text{vertices} - \#\text{edges} + \#\text{sides} = 1 \quad (+\text{ back side}) = 2$$