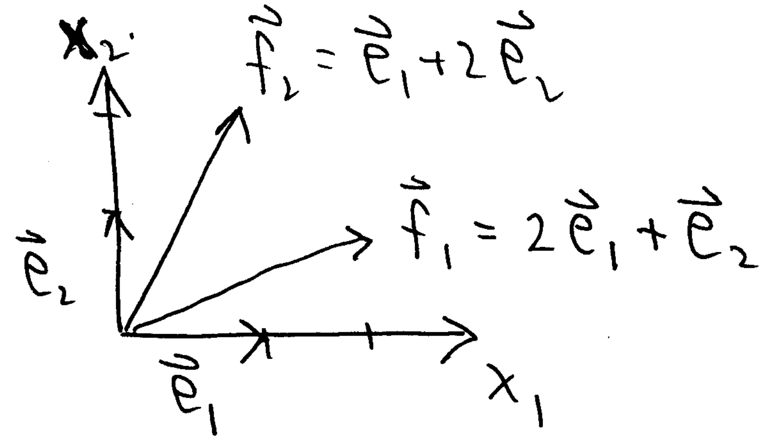
We want to define a linear transformation from vector space $\vec{V}$ to vector space $\vec{W}$.

To start: consider $T: \mathbb{R}^2 \to \mathbb{R}^2$



$T(\alpha\vec{v}_1 + \beta\vec{v}_2)$

$= \alpha T\vec{v}_1 + \beta T\vec{v}_2$

Now vectors in $\mathbb{R}^2$ can be named relative to any basis. Let $\vec{e}_1 = \overrightarrow{(1,0)}$, $\vec{e}_2 = \overrightarrow{(0,1)}$ denote the stanard basis, & let vector $\overrightarrow{(a,b)}$ name the vector that points to (a,b), the pt.

- A matrix $\underset{2\times 2}{A}$ then can describe how the components of $\vec{v}$ rel to $(\vec{e}_1, \vec{e}_2)$ basis get mapped to components of $T\vec{v}$ relative to $(\vec{e}_1, \vec{e}_2)$. The matrix that represents a transformation then depends on the basis

- Eg: assume $A = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$ represents a transformation T relative to basis $\{\vec{e}_1, \vec{e}_2\}$. Then

$$T(x_1\vec{e}_1 + x_2\vec{e}_2) = y_1\vec{e}_1 + y_2\vec{e}_2$$

where

$$\begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

In particular: $\begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow T(\vec{e}_1) = \vec{e}$

$$\begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \Rightarrow T(\vec{e}_2) = -\vec{e}_1 + 2\vec{e}_2$$

Conversely: assume

$$T(\vec{e}_1) = a\vec{e}_1 + b\vec{e}_2$$

$$T(\vec{e}_2) = c\vec{e}_1 + d\vec{e}_2$$

Then the matrix A that represents T in the $\{\vec{e}_1, \vec{e}_2\}$ basis is

$$A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}a\\b\end{pmatrix}, \; A\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}c\\d\end{pmatrix} \Rightarrow A = \begin{bmatrix}a & c\\ b & d\end{bmatrix}$$

Q: if we change basis to $(\vec{f}_1, \vec{f}_2)$, how does the matrix A that represents T change? I.e., assume $A = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$ represents T in basis $\{\vec{e}_1, \vec{e}_2\}$. What matrix represents T rel to $\{\vec{f}_1, \vec{f}_2\}$ where

$$\vec{f}_1 = \vec{e}_1 + 2\vec{e}_2$$
$$\vec{f}_2 = 2\vec{e}_2 + \vec{e}_1 \quad ?$$

① Change of basis matrix:

$$\begin{pmatrix} \vec{f}_1 \\ \vec{f}_2 \end{pmatrix} = \underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}}_{B} \begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \end{pmatrix}$$

$$B^{-1} = \frac{1}{-3}\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} \qquad \begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \end{pmatrix} = \frac{1}{-3}\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}\begin{pmatrix} \vec{f}_1 \\ \vec{f}_2 \end{pmatrix}$$

(2) Use B^{-1} to map comp's of $\vec{v}$ wrt $(\vec{f}_1, \vec{f}_2)$ over to comp's of $\vec{v}$ wrt $(\vec{e}_1, \vec{e}_2)$, then A to do the mapping T wrt $(\vec{e}_1, \vec{e}_2)$, then B to map comps back to $(\vec{f}_1, \vec{f}_2)$

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = B \quad A \quad B^{-1} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Comp's of $T\vec{v}$ wrt $\vec{f}_1, \vec{f}_2$

Comp's of $T\vec{v}$ wrt $(\vec{e}_1, \vec{e}_2)$

Comp's of $\vec{v}$ wrt $(\vec{e}_1, \vec{e}_2)$

comp's of $\vec{v}$ wrt $(\vec{f}_1, \vec{f}_2)$

For our example: if $A = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$ represents T wrt $(\vec{e}_1, \vec{e}_2)$, then

$$BAB^{-1} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} \left(-\frac{1}{3}\right) \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$$

represents T wrt $(\vec{f}_1, \vec{f}_2)$.

- More generally: if $T : V \to W$ is a linear transformation (eg $T : \mathbb{R}^n \to \mathbb{R}^m$).

If $\underset{m \times n}{A}$ represents T wrt $(\vec{e}_1, \dots, \vec{e}_n)$ – basis and

$$\begin{pmatrix} \vec{f}_1 \\ \vdots \\ \vec{f}_n \end{pmatrix} = B \begin{pmatrix} \vec{e}_1 \\ \vdots \\ \vec{e}_n \end{pmatrix}$$

Then BAB^{-1} represents T wrt $(\vec{f}_1 \cdots \vec{f}_n)$

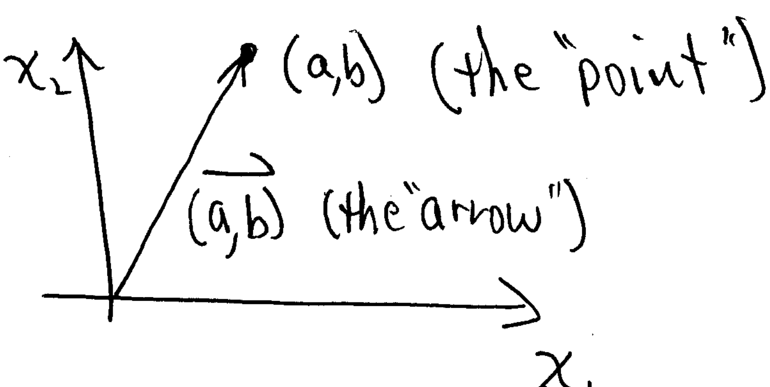
Matrices as linear transformations:

$$\underset{(m\times n)}{A}\,\underset{(n\times 1)}{x} = \underset{(m\times 1)}{b}$$

View matrix A as mapping a vector in $\mathbb{R}^n$ to a vector in $\mathbb{R}^m$: $x \longmapsto b$

- Really: A tells how the components of x relative to the standard basis in $\mathbb{R}^n$ gets mapped to components b relative to standard basis in $\mathbb{R}^n$

- __Ex__: $\mathbb{R}^2$: Let $\vec{e}_1 = \overrightarrow{(1,0)}$, $\vec{e}_2 = \overrightarrow{(0,1)}$ denote the standard basis. Let $\vec{v} = \overrightarrow{(a,b)}$ denote a vector in $\mathbb{R}^2$



- Note: distinction between the <u>point</u> (a,b), the vector $\overrightarrow{(a,b)}$ and the components of $\overrightarrow{(a,b)}$ relative to $(\vec{e}_1, \vec{e}_2)$:

$$\overrightarrow{(a,b)} = a\vec{e}_1 + b\vec{e}_2$$

- $A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\ c_1\\ 1\end{pmatrix} = \begin{pmatrix}0\\2\end{pmatrix}$

$A\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\ c_2\\ 1\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix}$

$\Rightarrow A = \begin{pmatrix}0 & -1\\ 2 & 1\end{pmatrix}$

Ex 1 : $T : \mathbb{R}^2 \to \mathbb{R}^2$ $\quad T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\-1\end{pmatrix}$

Find A. $\quad T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$

Soln : $A = \begin{bmatrix} | & | \\ c_1 & c_2 \\ | & | \end{bmatrix}$ $\qquad A \underset{\substack{\uparrow \\ \text{comps} \\ \text{of } v}}{x} = \underset{\substack{\uparrow \\ \text{comps} \\ Tv}}{y}$

$$A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\-1\end{pmatrix} \iff \begin{bmatrix} | & | \\ c_1 & c_2 \\ | & | \end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}2\\-1\end{bmatrix}$$

$$\iff \begin{bmatrix} | \\ c_1 \\ | \end{bmatrix} = \begin{bmatrix}2\\-1\end{bmatrix}$$

$$A\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} \iff \begin{bmatrix} | & | \\ c_1 & c_2 \\ | & | \end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$$

$$\iff \begin{bmatrix} | \\ c_2 \\ | \end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$$

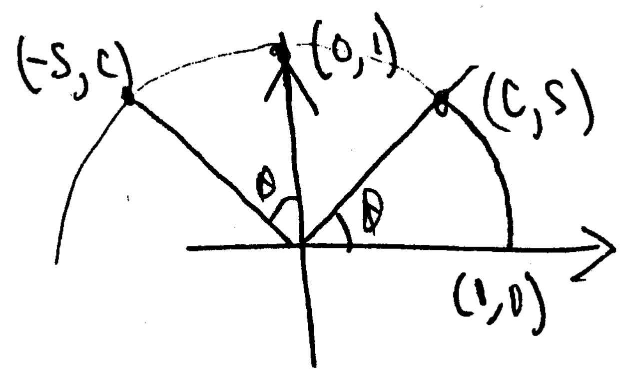
$$\Rightarrow A = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$$

Ex②: Find the matrix that represents rotation by angle θ in xy-plane.

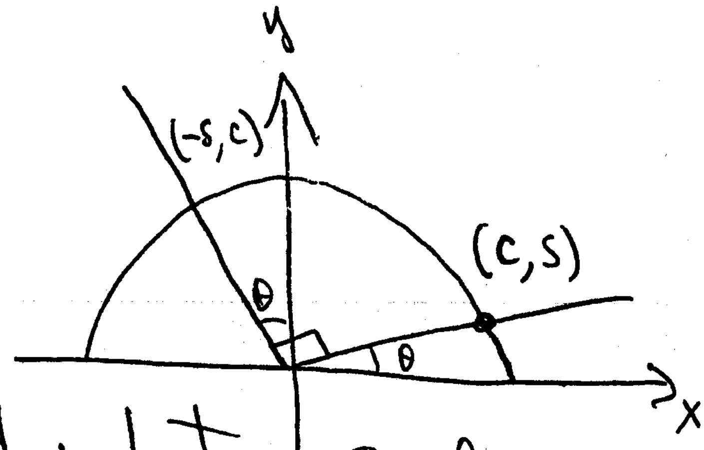
Soln: Look for 2x2 matrix $A(\theta)$ s.t.

$$A(\theta)\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}c\\s\end{pmatrix}$$

$$A(\theta)\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-s\\c\end{pmatrix}$$



$$A(\theta) = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \checkmark$$



Ex③: Find the matrix that rotates $\theta + \varphi$

Ans: $$\underbrace{\begin{bmatrix} c(\varphi) & -s(\varphi) \\ s(\varphi) & c(\varphi) \end{bmatrix} \underbrace{\begin{bmatrix} c(\theta) & -s(\theta) \\ s(\theta) & c(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}}_{\text{rotate } \theta}}_{\text{rotate } \varphi}$$

Check

$$\begin{bmatrix} c(\varphi) & -s(\varphi) \\ s(\varphi) & c(\varphi)\end{bmatrix}\begin{bmatrix} c(\theta) & -s(\theta) \\ s(\theta) & c(\theta)\end{bmatrix} = \begin{bmatrix} c(\varphi)c(\theta)-s(\varphi)s(\theta), & -c(\varphi)s(\theta)-s(\varphi)c(\theta) \\ s(\varphi)c(\theta)+c(\varphi)s(\theta), & -s(\varphi)s(\theta)+c(\varphi)c(\theta)\end{bmatrix}$$

$$= \begin{bmatrix} c(\varphi+\theta) & -s(\varphi+\theta) \\ s(\varphi+\theta) & c(\varphi+\theta)\end{bmatrix} \checkmark$$

~~In general: If A represents T and B represents S & T∘S~~

In general: $S \circ T = S(Tv)$ is epresented by $B \cdot A$ if A reps T & B rep's A ✓

Ex ③: Find matrix A that projects onto line $\angle\theta$ in $\mathbb{R}^2$

soln: Find image of basis $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$

$$A\begin{pmatrix}1\\0\end{pmatrix} = c\begin{pmatrix}c\\s\end{pmatrix} \qquad A\begin{pmatrix}0\\1\end{pmatrix} = s\begin{pmatrix}c\\s\end{pmatrix}$$

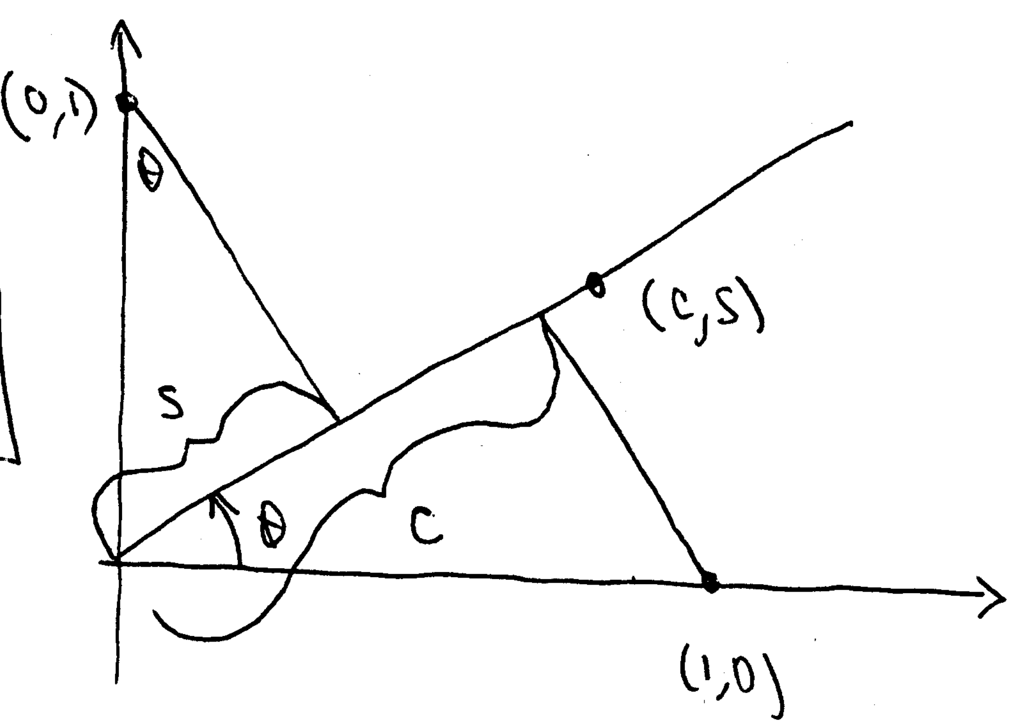
$$A = \begin{pmatrix}cc & sc\\ cs & ss\end{pmatrix} \equiv P(\theta)$$

check: $|A| = ccss - sccs = 0$ ✓

heck: $P^2 = P$

$$= \begin{bmatrix} c^2c^2 + c^2s^2, & c^2cs + s^2sc \\ c^2cs + s^2cs, & c^2s^2 + s^2s^2 \end{bmatrix}$$

$$\begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix} ✓$$



$s^2 = 1$

Ex: $\frac{dx}{dt} = \begin{bmatrix} \lambda_0 & 1 \\ 0 & \lambda_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\quad x(0) = x_0$

Soln: $|A - \lambda \underline{I}| = \begin{vmatrix} \lambda_0 - \lambda & 1 \\ 0 & \lambda_0 - \lambda \end{vmatrix} = (\lambda_0 - \lambda)^2 - 0 = (\lambda - \lambda_0)^2$

$\lambda = \lambda_0$

"double root"

$\begin{bmatrix} \lambda_0 - \lambda_0 & 1 \\ 0 & \lambda_0 - \lambda_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow x_2 = 0$

$\uparrow$

$R_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

One Soln: $x_1(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{\lambda_0 t}$: $x_2(t) = \begin{pmatrix} t e^{\lambda_0 t} \\ e^{\lambda_0 t} \end{pmatrix}$

Need another to ~~solve~~ meet initial condt.

Another one: $\begin{pmatrix} t e^{\lambda_0 t} \\ e^{\lambda_0 t} \end{pmatrix}' = \begin{pmatrix} e^{\lambda_0 t} + \lambda_0 t e^{\lambda_0 t} \\ \lambda_0 e^{\lambda_0 t} \end{pmatrix} = \begin{bmatrix} \lambda_0 & 1 \\ 0 & \lambda_0 \end{bmatrix} \begin{bmatrix} t e^t \\ e t \end{bmatrix}$ ✓

"Can reduce general resonant case to this one!