§2.6 Linear Transformations:

Defn: $T: V \to W$ between vector spaces is linear trans if

$$T(cu+dv) = cTu + dTv$$

"Can pass T thru add/scal mult" (Always assume: V, W are finite-

• **Ex**: an $m \times n$ matrix defines a lin trans from $\mathbb{R}^n \to \mathbb{R}^m$:

$$\underset{\substack{(m\times 1)\\ \uparrow \\ \text{in } \mathbb{R}^m}}{Tx} = \underset{(m\times n)}{A}\,\underset{\substack{(n\times 1)\\ \uparrow \\ \text{in } \mathbb{R}^n}}{x}$$

check: $A(cx+dy) = cAx + dAy$ ✓

Thm: Every linear trans. can be represented by a matrix, but the matrix depends on the choice of basis

Ex: $T: V \to W$. Let $\{v_1, \dots, v_n\}$ basis for V & $\{w_1, \dots, w_m\}$ basis for W. Then

$$Tv_1 = \sum_{i=1}^{m} w_i a_{i1} \quad (Tv_1 \in W\,!)$$

(*)

$$\vdots$$

$$Tv_n = \sum_{i=1}^{m} w_i a_{in} \quad (Tv_n \in W\,!)$$

Thus for any $v \in V$, $v = x_1 v_1 + \cdots + x_n v_n$ $\underset{\sim}{x} \in \mathbb{R}^n$

$$\left.\begin{matrix} x_1 Tv_1 & & \sum_{i=1}^{m} w_i a_{i1} x_1 \\ + & & + \\ \vdots & = & \vdots \\ + & & + \\ x_n Tv_n & & \sum_{i=1}^{m} w_i a_{in} x_n \end{matrix}\right\} (w_1, \dots, w_m)\, A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$A = (a_{ij})$

Notation: Bases $\{v_1 \cdots v_n\}$; $\{w_1, \ldots, w_m\}$

$$\underset{\sim}{v} = (v_1, \ldots, v_n) \in V^n$$

$$\underset{\sim}{w} = (w_1, \ldots, w_m) \in W^n$$

Components $\underset{\sim}{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$

$$\underset{\sim}{y} = (y_1, \ldots, y_m) \in \mathbb{R}^m$$

<u>Conclude</u>: if $v = \sum_{i=1}^{n} x_i v_i = \underset{\sim}{v}^T \cdot \underset{\sim}{x}$

$(v_1 \cdots v_n) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

Then

$$T(\underset{\sim}{v}^T \cdot \underset{\sim}{x}) = W^T \cdot A \underset{\sim}{x} \qquad A = (a_{ij})$$

"T sends the $\underset{\sim}{x}$-linear comb of $\underset{\sim}{v}$" TO "The $A\underset{\sim}{x}$-linear comb of $\underset{\sim}{w}$"

"$A = (a_{ij})$ is the matrix representation of T in terms of bases $\underset{\sim}{v}$, $\underset{\sim}{w}$"

$$\underset{(m\times 1)}{\underset{\sim}{y}} = \underset{(m\times n)}{A} \underset{(n\times 1)}{\underset{\sim}{x}}$$

comp's of Tv rel to $\underset{\sim}{w}$

comp's of v rel to $\underset{\sim}{v}$

<u>Corollary</u>: A linear transformation is determined by where it sends a <u>basis</u>

<u>Important</u>: The matrix $A = (a_{ij})_{m\times n}$ depends on choice of bases!

☒ Linear Transformations $\mathbb{R}^n \to \mathbb{R}^m$

- Matrix $A_{(m\times n)}$ <u>determines</u> a linear Transf.

$$y = Ax$$

$$(y_1, \dots, y_m) = A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

- <u>Actually</u>: This is the matrix associated with the standard bases $(e_1, \dots, e_n)$ on $\mathbb{R}^n$ & $(e_1, \dots, e_m) \in \mathbb{R}^m$

Eg: $T(\underset{\sim}{e}\cdot\underset{\sim}{x}) = \underset{\sim}{e}\cdot A\cdot\underset{\sim}{x}$

$\Leftrightarrow$ $\underbrace{T\left(\sum_{i=1}^{n} x_i e_i\right)}_{\underset{\sim}{e}^T\cdot\underset{\sim}{x}} = (e_1,\dots,e_m)\, \underbrace{A\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}$

$\underset{\sim}{x}\in\mathbb{R}^n$ give comps of v wrt standard basis $\{e_1,\dots,e_n\}$

$A\underset{\sim}{x}$ gives comp of Tv in terms of standard basis $\{e_1,\dots,e_m\}$

$\Leftrightarrow$ $Ax = y$

comps of v

comps Tv

Ex 1: $T : \mathbb{R}^2 \to \mathbb{R}^2$ $\quad T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\-1\end{pmatrix}$

Find A. $\quad T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$

Soln: $A = \begin{bmatrix} | & | \\ c_1 & c_2 \\ | & | \end{bmatrix}$ $\qquad A \underset{\substack{\uparrow \\ \text{comps} \\ \text{of } v}}{x} = \underset{\substack{\uparrow \\ \text{comps} \\ Tv}}{y}$

$$A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\-1\end{pmatrix} \iff \begin{bmatrix} | & | \\ c_1 & c_2 \\ | & | \end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}2\\-1\end{bmatrix}$$

$$\iff \begin{bmatrix} | \\ c_1 \\ | \end{bmatrix} = \begin{bmatrix}2\\-1\end{bmatrix}$$

$$A\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} \iff \begin{bmatrix} | & | \\ c_1 & c_2 \\ | & | \end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$$

$$\iff \begin{bmatrix} | \\ c_2 \\ | \end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$$

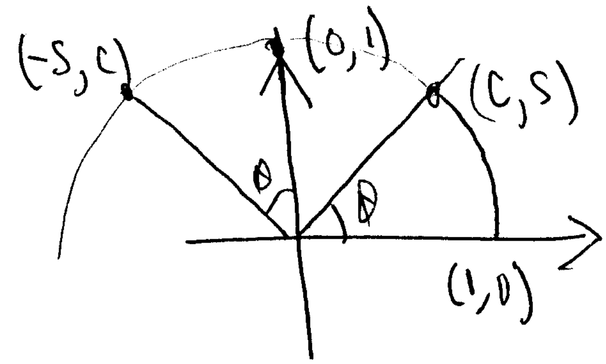
$$\Rightarrow A = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$$

Ex 2 : Find the matrix that represents rotation by angle θ in xy-plane.

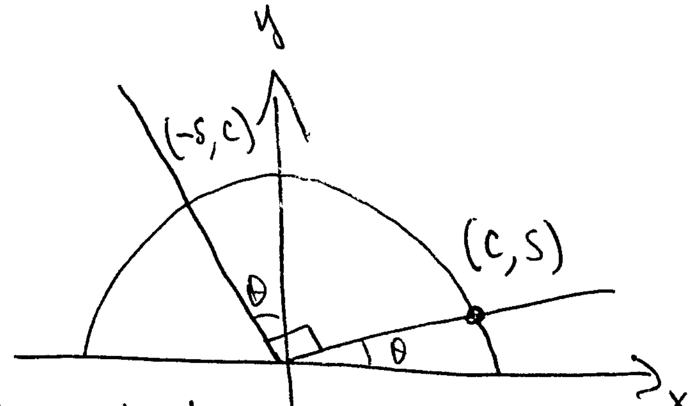
Soln : Look for 2x2 matrix $A(\theta)$ st

$$A(\theta)\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}c\\s\end{pmatrix}$$

$$A(\theta)\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-s\\c\end{pmatrix}$$



$$A(\theta) = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$$ ✓



Ex 3 : Find the matrix that rotates $\theta + \varphi$

Ans : $$\underbrace{\begin{bmatrix} c(\varphi) & -s(\varphi) \\ s(\varphi) & c(\varphi) \end{bmatrix} \underbrace{\begin{bmatrix} c(\theta) & -s(\theta) \\ s(\theta) & c(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}}_{\text{rotate } \theta}}_{\text{rotate } \varphi}$$

Check

$$\begin{bmatrix} c(\varphi) & -s(\varphi) \\ s(\varphi) & c(\varphi) \end{bmatrix}\begin{bmatrix} c(\theta) & -s(\theta) \\ s(\theta) & c(\theta) \end{bmatrix} = \begin{bmatrix} c(\varphi)c(\theta)-s(\varphi)s(\theta) \, , & -c(\varphi)s(\theta)-s(\varphi)c(\theta) \\ s(\varphi)c(\theta)+c(\varphi)s(\theta) \, , & -s(\varphi)s(\theta)+c(\varphi)c(\theta) \end{bmatrix}$$

$$= \begin{bmatrix} c(\varphi+\theta) & -s(\varphi+\theta) \\ s(\varphi+\theta) & c(\varphi+\theta) \end{bmatrix}$$

~~In general: If A represents T and B represents S & T∘S~~

- In general : $S \circ T = S(Tv)$ is represented by $B \cdot A$ if A reps T & B rep's A

Ex 3: Find matrix A that projects onto line $\angle\theta$ in $\mathbb{R}^2$

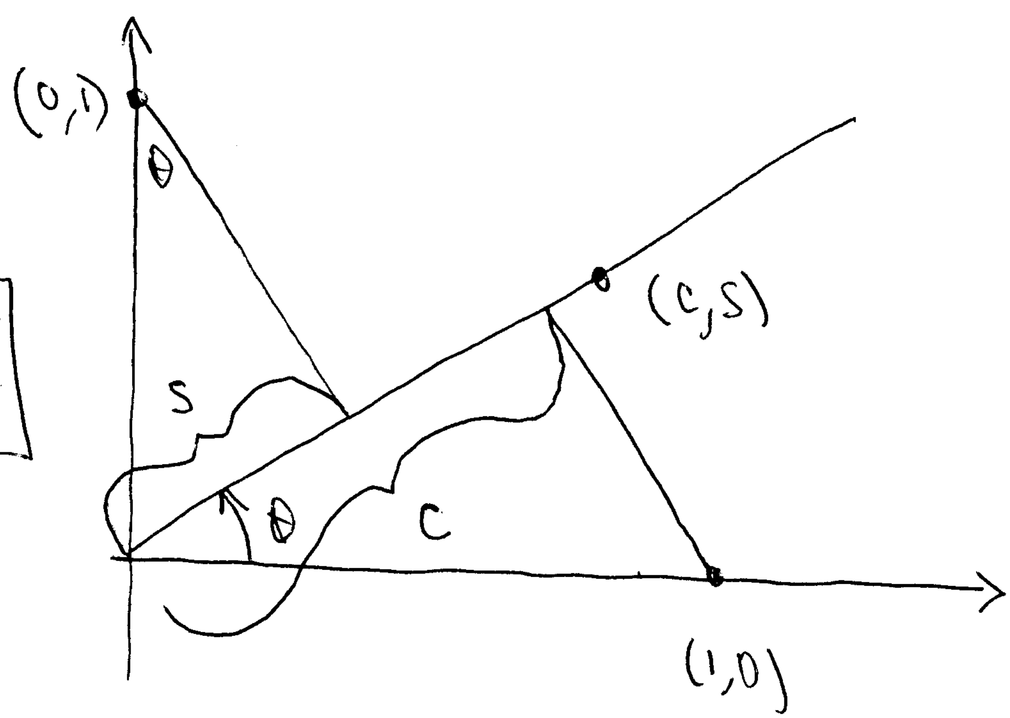
Soln: Find image of basis $\begin{pmatrix}1\\0\end{pmatrix}$, $\begin{pmatrix}0\\1\end{pmatrix}$

$$A\begin{pmatrix}1\\0\end{pmatrix} = c\begin{pmatrix}c\\s\end{pmatrix} \qquad A\begin{pmatrix}0\\1\end{pmatrix} = s\begin{pmatrix}c\\s\end{pmatrix}$$

$$A = \begin{pmatrix}cc & sc\\ cs & ss\end{pmatrix} \equiv P(\theta)$$

check: $|A| = ccss - sccs = 0$ ✓

check: $P^2 = P$



$$P^2 = \begin{bmatrix} c^2c^2 + c^2s^2, & c^2 s + s^2 sc \\ c^2 cs + s^2 cs, & c^2s^2 + s^2s^2 \end{bmatrix}$$

$$= \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}$$ ✓

$c^2 + s^2 = 1$