Hermitian matrices:

- In math physics, complex valued matrices A are important —

<u>Defn</u>: $A_{n\times n}$ (complex) is Hermitian Symmetric if

$$\langle Ax, y\rangle = \langle x, Ay\rangle \iff \bar{A}^T = A$$

$\forall$ complex x, y $\quad \langle x,y\rangle = \bar{x}\cdot y = \bar{x}^T y$

dot $\qquad$ matrix

<u>Defn</u>: $A^H \equiv \bar{A}^T$

<u>Thm</u>: If A is Hermitian, then evals of A are real, & $\exists$ o.n. basis of (complex) eigenvectors

Pf. (exactly same proof!)

Real evals: $\langle Ax, x\rangle = \langle x, Ax\rangle$

$$\langle \lambda x, x\rangle = \langle x, \lambda x\rangle$$

$$\bar{\lambda}\cancel{(\bar{x}\cdot x)} = \cancel{(\bar{x}\cdot x)}\lambda$$

$$\bar{\lambda} = \lambda \checkmark$$

$x_1 \perp x_2$: $\langle Ax_1, x_2\rangle = \langle x_1, Ax_2\rangle$

$$\langle \lambda_1 x_1, x_2\rangle = \langle x_1, \lambda_2 x_2\rangle$$

$$\lambda_1 \underbrace{\cancel{(\bar{x}_1\cdot x_2)}}_{0} = \lambda_2 \underbrace{\cancel{(\bar{x}_1\cdot x_2)}}_{0}$$

$\lambda_1 \neq \lambda_2 \Rightarrow \bar{x}_1 \cdot x_2 = 0$ ✓

<u>That</u> x_i can be completed to ON-basis when A has repeated evals follows by same appl. of Shur's lemma .... (omit)

<u>Thm</u>: A Hermitian implies

$$U^{-1} A U = \Lambda$$

where U is <u>unitary</u>

Pf. Choose $U = \begin{bmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{bmatrix}$ (complex)

<u>Defn</u>: U unitary if colms of U are o.n. in complex sense

<u>Thm</u>: U unitary $\Rightarrow$ ① $U^{-1} = \bar{U}^T = U^H$

and ② $|\lambda| = 1$ $\forall$ eval of U

Pf. ① $\underbrace{\begin{bmatrix} - \bar{x}_1 - \\ \vdots \\ - \bar{x}_n - \end{bmatrix}}_{U^H} \underbrace{\begin{bmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{bmatrix}}_{U} = (\bar{x}_i \cdot x_j) = \delta_{ij}$

$\uparrow$ dot

① $\Rightarrow \|Ux\|^2 = (\overline{Ux})^T Ux = \bar{x}^T \underbrace{\bar{U}^T U}_{id} x = \bar{x}^T x = |x|^2$

(length preserving)

For 2 Use: If $Ux = \lambda x$, then

$$\cancel{\|Ux\|} = \|\lambda x\| = |\lambda| \cancel{\|x\|}$$

$$\|Ux\| = \|x\|$$

$$\Rightarrow \boxed{1 = |\lambda|} \checkmark$$

In case $\underset{m\times n}{A}$ complex, same argument leads to SVD

$$A = U \Sigma \bar{V}^T$$

$\{u_i\}$ ON basis of $A\bar{A}^T$, $\{v_i\}$ ON basis for $\bar{A}^T A$