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FINAL EXAM

Math 167
Temple-F08
Problem 1. (30pts) Let $A$ be a real $n \times n$ symmetric matrix.

(a) Prove that $A$ has only real eigenvalues.

(b) Assuming that $A$ has $n$ distinct eigenvalues, prove that $A$ has an orthonormal basis of eigenvectors.
Problem 1. (Continued)
Problem 2. (25pts) Let

\[ A = \begin{bmatrix}
\frac{1}{2} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & 0
\end{bmatrix}. \]

(a) Find the eigenvalues of \( A \).
(b) Find an orthonormal basis of eigenvectors of \( A \).
(c) Find an \( 2 \times 2 \) orthogonal matrix \( S \) such that \( A = SDS^T \), where \( D \) is a diagonal matrix.
(d) Find \( \lim_{n \to \infty} A^n \).
Problem 2. (Continued)
Problem 3. (20pts) Let $A$ and $B$ be $m \times n$ matrices with entries $a_{ij}$ and $b_{ij}$, respectively. Prove that if $Av_i = Bv_i$ for some basis\{v_1, \cdots, v_n\} of $\mathbb{R}^n$, then $a_{ij} = b_{ij}$ for every $i, j = 1 \cdots n$. 
Problem 3. (Continued)
Problem 4. (25pts) Assume that the populations $y_1$ and $y_2$ of two interacting species of animals evolves according to the equation

$$ y'(t) = \begin{bmatrix} y'_1(t) \\ y'_2(t) \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} y_1(t) - a \\ y_2(t) - b \end{bmatrix}. \quad (1) $$

(a) Show that the constant populations $y_1 = a$, $y_2 = b$ solves (1).

(b) Show that $y = \begin{bmatrix} a \\ b \end{bmatrix}$ is a stable rest point by finding a formula for the general solution $y(t)$ of (1) and showing that

$$ \lim_{t \to \infty} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} $$

(Hint: Let $x_1 = y_1 - a$, $x_2 = y_2 - b$, and find the eigen-solutions of $x' = Ax.$)
Problem 4. (Continued)
**Problem 5. (25pts)** Assume that $A$ is a square $n \times n$ matrix of real numbers, and assume that the (symmetric) product $A^T A$ is **positive definite** in the sense that $x^T A^T A x > 0$, (strictly positive!), for every vector $x \in \mathbb{R}^n$.

(a) Prove that there exists an orthonormal basis $\{v_1, ..., v_n\}$ of $\mathbb{R}^n$ and $n$ non-negative numbers $\lambda_i > 0$ such that $A^T A v_i = \lambda_i v_i$.

(b) Prove that $u_i = Av_i$ is an **orthogonal** basis of eigenvectors of $AA^T$ in the sense that $\langle u_i, u_j \rangle = u_i \cdot u_j = 0$, (i.e. $u_i \perp u_j$), when $i \neq j$. 
Problem 5. (Continued)
Problem 6. (25pts) Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Compute $A^n$, and use this to derive the matrix $e^A$ directly from the definition.
Problem 6. (Continued)
Problem 7. (25pts)

(a) If \( x = c_1 u_1 + \cdots + c_n u_n \) gives vector \( x \) in terms of a given orthonormal basis \( \{ u_1, \ldots, u_n \} \), derive a formula for \( c_i \).

(b) Use Gramm-Schmidt to construct an orthonormal basis \( u_1, u_2, u_3 \) for \( \mathbb{R}^3 \) from the basis

\[
\begin{bmatrix}
1 \\
0 \\
1 
\end{bmatrix}, \quad
\begin{bmatrix}
1 \\
0 \\
0 
\end{bmatrix}, \quad
\begin{bmatrix}
1 \\
0 \\
1 
\end{bmatrix}.
\]
Problem 7. (Continued)
Problem 8. (25pts) Let \( A_{m \times n} \) satisfy \( m > n \), and assume the columns of \( A \) form a basis for \( R^n \). For \( b \in R^m \), let \( A \hat{x} \) denote the element of \( Col(A) \) closest to \( b \).

(a) Show that \( \hat{x} = (A^T A)^{-1} A^T b \).

(b) Find the line \( b = C + Dt \) that best fits the data points \((t_1, b_1) = (1, -1); \ (t_2, b_2) = (-1, 1); \ (t_3, b_3) = (1, 2)\) in the sense that \( \sum_{i=1}^{3} (b_i - (C + Dt_i))^2 \) is minimized.
Problem 8. (Continued)