Name:

Student ID:
Section:

1	2	3	4	5	6	Total

MIDTERM EXAM
Math 167
Temple-W10

Problem 1. (20pts) True or False:

(Here A is an arbitrary $m \times n$ matrix $Col(A)$ denotes the Column Space of A, $Row(A)$ the Row Space, and $Ker(A)$ the Kernel.)

F (a) $dim\,\{Row(A)\} = dim\,\{Col(A)\}$ except when $m < n$.

T (b) If $m = n$ and $Det(A) \neq 0$, then $Ax = b$ has a unique solution for every $b \in \mathcal{R}^n$.

T (c) If $m < n$, and A has maximal rank, then we must have $dim\,\{Ker(A)\} = n - m$.

T (d) If $A = LDL^T$ where L is lower triangular with 1's on the diagonal and D is diagonal, then A is symmetric.

T (e) Let $E = E_{ij}(a)$ denote the matrix obtained from the $m \times m$ identity matrix by putting a in the (i, j)-entry. Then the matrix multiplication $E \cdot A$ makes sense, and the effect is to add a times the j'th row of A to the i'th row of A.

F (f) Let $P = P_{ij}$ denote the matrix obtained from the $m \times m$ identity matrix by interchanging the i'th and j'th rows. Then the matrix multiplication $P \cdot A$ makes sense, and the effect is to "Put"the i'th row of A into the j'th column.

F (g) In general it takes more operations to do back substitution than it does to do Gaussian elimination.

F (h) Let u and v be column vectors in $\mathcal{R}^n$. Then the rows of the rank-1 matrix $u \cdot v^t$ are all multiples of u.

F (i) If $m < n$, then A cannot have a right inverse.

F (j) The solution space of all x such that $Ax = b$ is always a vector space.

Problem 2. (15pts) Use elementary matrices to find matrices L and U^*, (L lower triangular with 1's on the diagonal, U^* upper triangular), such that $A = LU$, assuming

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 4 & 1 \\ -2 & -6 & 2 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

Problem 3. (15pts) Let $A = LU$ where

$$L = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

Solve $Ax = b$ for $x = (x_1, x_2, x_3)$ by the **most efficient method**.

Solution: Most efficient way: Solve $Lc = b$ for c by forward substitution, then solve $Ux = c$ by backward substitution.

$Lc = b$ gives

$$\begin{aligned} c_1 &= -1 \\ c_2 &= \frac{1}{2}(2 - c_1) = 3/2 \\ c_3 &= (-1 - c_1 + c_2) = 3/2 \end{aligned}$$

$Ux = c$ gives

$$\begin{aligned} x_3 &= 3/2 \\ x_2 &= -(3/2 + x_3) = 3 \\ x_1 &= \frac{1}{2}(-1 - x_2 + x_3) = -5/4 \end{aligned}$$

Problem 4. (20pts) Consider the problem $Ax = 0$ where

$$A = \begin{bmatrix} 2 & 1 & 0 & 2 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

(a) Find the *pivots* of A, and the rank of A.

Solution: $p_1 = 2$, $p_2 = -3$.

(b) Determine the *pivot variables* and the *free variables* in x.

Solution: x_1, x_3 are the pivot variables, x_2, x_4 are the free variables.

(c) Find matrices D and R such that $A = DR$ where D is diagonal and R is the reduced row eschelon form of A.

Solution:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & 0 & 1 \\ 0 & 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(d) Find a basis for $Row(A)$.

Solution: A basis for $Row(A)$ is: $\{(2,1,0,2),(0,0,-3,1)\}$

(e) Find a basis for $Ker(A)$.

Solution: $x \in Ker(A)$ iff $x_1 + \frac{1}{2}x_2 + x_4 = 0$ and $x_3 - \frac{1}{2}x_2 + x_4 = 0$, or

$$x = \begin{bmatrix} -\frac{1}{2}x_2 - x_4 \\ x_2 \\ \frac{1}{2}x_2 + x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -\frac{1}{2} \\ 1 \\ \frac{1}{2} \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

So a basis for the Ker(A) is: $\{(-1,2,1,0),(-1,0,1,1)\}$

Problem 5. (15pts) Find the matrix A that represents a linear transformation $T : \mathcal{R}^3 \to \mathcal{R}^5$ in terms of the standard basis $\mathbf{e}_i$, (the vector with 1 in the i'th position and zeros elsewhere), if

$$\begin{aligned} \mathbf{e}_1 &\to \mathbf{e}_2 - \mathbf{e}_1 \\ \mathbf{e}_2 &\to \mathbf{e}_4 - 3\mathbf{e}_2 \\ \mathbf{e}_3 &\to -\mathbf{e}_5 + 2\mathbf{e}_3. \end{aligned}$$

Find the dimension of the kernel of T.

Solution:

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

The rank of the matrix is 3, so the $dim[Ker(T)]$=3-3=0.

Problem 6. (15pts) Let $\{v_1, ..., v_k\}$ be a finite set of vectors in a vector space V.

(a) Complete the definition:

Solution:
$Span\,\{v_1, ..., v_k\} = \{v \in V : \; v = c_1v_1 + \cdots + c_kv_k$ for some $c_i \in \mathcal{R}\}$

(b) Define what it means for $\{v_1, ..., v_k\}$ to be *linearly dependent.*

Solution: $\{v_1, ..., v_k\}$ are linearly dependent if there exist real numbers $c_1, \cdots, c_k$, not all zero, such that $c_1v_1 + \cdots + c_kv_k = 0$.

(c) Prove that if $\{v_1, ..., v_k\}$ are linearly dependent, then at least one vector can be removed from the list without changing $Span\,\{v_1, ..., v_k\}$.

Solution: Since by (b), $c_1v_1 + \cdots + c_kv_k = 0$ for some $c_i \neq 0$, you can solve for that v_i obtaining v_i as a linear combination of the others. Omitting this one from the set, every linear combination of $v_1, ..., v_k$ can be written as a linear combination of $v_1, ..., v_k$ *excluding* v_i, by simply substituting v_i as a linear combination of the others.