FINAL EXAM
Math 167
Temple-Winter 2010

−Print your name, section number and put your signature on the upper right-hand corner of this exam. Write only on the exam.

−Show all of your work, and justify your answers.

SCORES

#1	#9
#2	#10
#3	#11
#4	#12
#5	#13
#6	#14
#7	#15
#8	

TOTAL:

1. (10 pts) Determine the following limits:

(a) (5 pts) $\lim_{x\to 3}\left(\frac{x^2-2x+3}{x-3}\right)^2$

$$\lim_{x\to 3}\left\{\left(\frac{x^2-2x+3}{x-3}\right)^2 = \left(\frac{(x-3)(x+1)}{x-3}\right)^2 = (x+1)^2\right\} = 16$$

(b) (5 pts) $\lim_{x\to 0}\frac{\sqrt{3}-\sqrt{x+3}}{4x}$

$$\begin{aligned}\lim_{x\to 0}\left\{\frac{\sqrt{3}-\sqrt{x+3}}{4x}\right. &= \frac{\sqrt{3}-\sqrt{x+3}}{4x}\left(\frac{\sqrt{3}+\sqrt{x+3}}{\sqrt{3}+\sqrt{x+3}}\right\} = \left.\frac{3-x-3}{4x(\sqrt{3}+\sqrt{x+3})}\right\} \\ &= \frac{-1}{4(\sqrt{3}+\sqrt{6})}\end{aligned}$$

2. (10 pts) Determine the following limits:

(a) (5 pts) $\lim_{x\to-\infty} \frac{x^{11}+3x-4}{5x^{11}-7}$

$$\lim_{x\to-\infty} \frac{x^{11}+3x-4}{5x^{11}-7} = \frac{1}{5}$$

(b) (5 pts) $\lim_{x\to\frac{\pi}{4}} \cos x \sin x$

$$\lim_{x\to\frac{\pi}{4}} \cos x \sin x = \cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right) = \frac{1}{2}$$

3. (16 pts) Let $f(x) = \sqrt{1-x^2}$, $\quad g(x) = \frac{1}{2+\cos x}$.

(a) (4 pts) Find the Domains of f and g and express them in set bracket notation.

The Domain of f is the set [-1,1]; the Domain of g is $(-\infty, \infty)$

(b) (4 pts) Find the (precise) Range of g and express it in set bracket notation.

The Range of g is the set $[\frac{1}{3}, 1]$.

(c) (4 pts) Find the composition function $(f \circ g)(x)$.

$$(f \circ g)(x) = f(g(x)) = \sqrt{1 - \left(\frac{1}{2+\cos x}\right)^2}$$

(d) (4 pts) Find the Domain of $(f \circ g)(x)$. (Justify)

All real numbers $(-\infty, +\infty)$ because Range of g is a subset of the Domain of f.

4. (13 pts) Consider the function $f(x) = \frac{3x+1}{x-2}$ with Domain $x \neq 2$.

(a) (7 pts) Find a formula for $f^{-1}(x)$.

$$y = \frac{1+2x}{x-3}$$

(b) (4 pts) Find the Domain of f^{-1}.

$$x \neq 3$$

(c) (2 pts) Evaluate $f^{-1}(1)$

$$f^{-1}(1) = \frac{1+2}{1-3} = -\frac{3}{2}.$$

5. (15 pts) Find the vertical asymptotes (you needn't graph the functions):

(a) (5 pts) $y = \sin x$

None

(b) (5 pts) $y = \frac{2x^2}{(x^2-4)(x-1)}$

$$x = \pm 2, 1$$

(c) (5 pts) $y = \tan x$

$$x = \frac{\pi}{2} + n\pi, \quad n = 0, \pm 1, \pm 2, \pm 3...$$

6. (20 pts) Differentiate: (Do not simplify.)

(a) (5 pts) $y = x^5 + 6x^3 - 144$

$y' = 5x^4 + 18x^2$

(b) (5 pts) $y = (2x^2 - 3)(x^{-2} + 1)$

$y' = (2x^2 - 3)(-2x^{-3}) + (4x)(x^{-2} + 1)$

(c) (5 pts) $y = \frac{\cos x}{\sin x}$

$y' = \frac{\sin x(-\sin x) - \cos x \cos x}{\sin^2 x}$

(d) (5 pts) y=$\cos(x^5 + 2x)$

$y' = -(5x^4 + 2)\sin(x^5 + 2x)$

7. (10 pts) Differentiate: $f(x) = \frac{\sin^2(x+\sqrt{x})}{x\cos x}$ (Do not simplify.)

$$f'(x) = \frac{x\cos x[2\sin(x+\sqrt{x})(1-\frac{1}{2}x^{-1/2})] - \sin^2(x+\sqrt{x})[\cos x - x\sin x]}{(x\cos x)^2}$$

8. (12 pts) Let $f(x) = x^2+5x$. Use the mathematical definition of derivative to derive a formula for $f'(x)$.

$\left(\text{Recall the mathematical definition : } f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.\right)$

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \left\{ \frac{f(x+h)-f(x)}{h} \right. \\
&= \left. \frac{(x+h)^2 + 5(x+h) - x^2 - 5x}{h} = 2x + h + 5 \right\} = 2x + 5.
\end{aligned}
$$

9. (16 pts) Consider the following complicated equation:

$$2xy^3 + 4x^2 - 2y^4 = 0.$$

(a) (8 pts) Use implicit differentiation to obtain a formula for $y' = \frac{dy}{dx}$ at a point (x, y) on the graph.

$$y' = -\frac{8x + 2y^3}{6xy^2 - 8y^3}$$

(b) (8 pts) Find the slope m and y-intercept b of the line tangent to the graph of f at the point $(1, -1)$.

$$m = -\frac{3}{7}, \quad b = -\frac{4}{7}$$

10. (20 pts) Let $y = f(x) = \frac{1}{4}x^4 - 2x^2 - 2$.

(a) (5 pts) Find the values of x that are *critical points.*

$$y' = x^3 - 4x = x(x^2 - 4) = 0,$$

Critical points at $x = 0, \pm 2$.

(b) (5 pts) Use the second derivative test to determine the values of x that are *relative maxima* and *relative minima.*

$$f''(x) = 3x^2 - 4,$$

so

$$f''(0) = -4 < 0, \quad f''(\pm 2) = (3)(4) - 4 > 0,$$

implying $x = 0$ is a relative max and $x = \pm 2$ are relative min's.

(c) (5 pts) Find the values of x that are *inflection points.*

$f''(x) = 3x^2 - 4 = 0$ implies $x = \pm\frac{2}{\sqrt{3}}$ are the inflection points.

(d) (5 pts) Use interval notation (brackets) to describe the set of x where f is *convex up.*

$f''(x) = 3x^2 - 4 > 0$ implies $x \in (-\infty, -\frac{2}{\sqrt{3}}) \cup (\frac{2}{\sqrt{3}}, +\infty)$.

11. (12 pts) Two ships pass near a lighthouse. Let $x(t)$ denote the distance (in miles) from the first ship to the lighthouse, and $y(t)$ the distance from the second ship to the lighthouse, and assume you know that at every time, $x^2 + y^3 = 17$. Find the speed $\frac{dy}{dt}$ when the first ship is 3 miles from the lighthouse moving at speed $\frac{dx}{dt} = 1$ mile per hour.

$$x(t)^2 + y(t)^3 = 17$$

holds at every time, so

$$2xx' + 3y^2y' = 0$$

holds at every time. We know $x = 3, x' = 1$ and $x^2+y^3 = 17$ gives $y = 2$. Thus

$$y' = \frac{-6}{3 \cdot 4} = -\frac{1}{2}.$$

12. (12 pts) The cost of material to build a rectangular vertical wall L feet long and H feet high is $2H^2 + 3L$ dollars. Find the dimensions of the wall of area 36 square feet that minimizes the cost.

$$C = 2H^2 + 3L, \quad \text{and} \quad L \cdot H = 36.$$

Solving $L = \frac{36}{H}$ and substituting gives

$$C = 2H^2 + \frac{3 \cdot 36}{H},$$

so

$$\frac{dC}{dH} = 4H - \frac{3 \cdot 36}{H^2} = 0$$

gives

$$H^3 = 27,$$

or $H = 3$, $L = 12$. Range of variables $(0, \infty)$ with $C = \infty$ at the endpoints confirms this is the minimum.

13. (12 pts) The radius r of a circle is increasing at a rate of 5 meters per second.

(a) (6 pts) Find a formula for the rate $\frac{dA}{dt}$ at which the area is increasing.

$$A = \pi r^2, \quad \frac{dA}{dt} = 2\pi r \frac{dr}{dt} = 10\pi r$$

(b) (6 pts) Use differentials to to give the estimate dA for the increase in area over time interval $dt = 2$ seconds when the radius is 10 meters.

$$dA = 10\pi r dr = 200\pi.$$

14. (10 pts) Sketch the graph of a function continuous except at $x = 2$, such that

$$\lim_{x\to 2-} = -1, \quad \lim_{x\to 2+} = +\infty, \quad f(2) = 1.$$

Graph comes in from the left to an open dot at $(2, -1)$; there is a closed dot at $(2, 1)$; and the graph goes off to $y = +\infty$ as it comes in to $x = 2$ from the right.

15. (12 pts) Find the point on the graph of $y = \sqrt{x}$, $x \geq 0$, closest to the point $(1, 0)$.

$$D = d^2 = (x-1)^2 + (y-0)^2, \quad y = \sqrt{x}$$

gives

$$D = (x-1)^2 + x.$$

Differentiating gives

$$\frac{dD}{dx} = 2(x-1) + 1 = 0,$$

giving a unique critical point. Plugging in for y gives critical point on graph as $x = \frac{1}{2}$, $y = \frac{1}{\sqrt{2}}$.

Now the distance $D \to \infty$ as $x \to +\infty$, and so the minimum distance must be at endpoint $x = 0$ or critical point $x = \frac{1}{2}$. Checking $D(0) = 1$ and $D(\frac{1}{2}) = \frac{1}{4} + \frac{1}{2} < 1$, it follows that the closest point must is the critical point $x = \frac{1}{2}$, $y = \frac{1}{\sqrt{2}}$.