FINAL EXAM
Math 167
Temple-Winter 2010

—Print your name, section number and put your signature on the upper right-hand corner of this exam. Write only on the exam.

—Show all of your work, and justify your answers.

SCORES

#1 #9
#2 #10
#3 #11
#4 #12
#5 #13
#6 #14
#7 #15
#8

TOTAL:
1. (10 pts) Determine the following limits:

(a) (5 pts) \( \lim_{x \to 3} \left( \frac{x^2 - 2x + 3}{x - 3} \right)^2 \)

\[
\lim_{x \to 3} \left( \frac{x^2 - 2x + 3}{x - 3} \right)^2 = \left( \frac{(x - 3)(x + 1)}{x - 3} \right)^2 = (x + 1)^2 \]

= 16

(b) (5 pts) \( \lim_{x \to 0} \frac{\sqrt{3} - \sqrt{x + 3}}{4x} \)

\[
\lim_{x \to 0} \frac{\sqrt{3} - \sqrt{x + 3}}{4x} = \frac{\sqrt{3} - \sqrt{x + 3}}{4x} \left( \frac{\sqrt{3} + \sqrt{x + 3}}{\sqrt{3} + \sqrt{x + 3}} \right) = \frac{3 - x - 3}{4x(\sqrt{3} + \sqrt{x + 3})}
\]

= \(-\frac{1}{4(\sqrt{3} + \sqrt{6})}\)
2. (10 pts) Determine the following limits:
   
   (a) (5 pts) \( \lim_{x \to -\infty} \frac{x^{11} + 3x - 4}{5x^{11} - 7} \)
   
   \[ \lim_{x \to -\infty} \frac{x^{11} + 3x - 4}{5x^{11} - 7} = \frac{1}{5} \]

   (b) (5 pts) \( \lim_{x \to \frac{\pi}{4}} \cos x \sin x \)
   
   \[ \lim_{x \to \frac{\pi}{4}} \cos x \sin x = \cos \left( \frac{\pi}{4} \right) \sin \left( \frac{\pi}{4} \right) = \frac{1}{2} \]
3. (16 pts) Let \( f(x) = \sqrt{1 - x^2}, \quad g(x) = \frac{1}{2 + \cos x}. \)

(a) (4 pts) Find the Domains of \( f \) and \( g \) and express them in set bracket notation.

The Domain of \( f \) is the set \([-1, 1]\); the Domain of \( g \) is \((\infty, \infty)\).

(b) (4 pts) Find the (precise) Range of \( g \) and express it in set bracket notation.

The Range of \( g \) is the set \([\frac{1}{3}, 1]\).

(c) (4 pts) Find the composition function \((f \circ g)(x)\).

\[
(f \circ g)(x) = f(g(x)) = \sqrt{1 - \left(\frac{1}{2 + \cos x}\right)^2}
\]

(d) (4 pts) Find the Domain of \((f \circ g)(x)\). (Justify)

All real numbers \((-\infty, +\infty)\) because Range of \( g \) is a subset of the Domain of \( f \).
4. (13 pts) Consider the function \( f(x) = \frac{3x+1}{x-2} \) with Domain \( x \neq 2 \).

(a) (7 pts) Find a formula for \( f^{-1}(x) \).

\[
y = \frac{1 + 2x}{x - 3}
\]

(b) (4 pts) Find the Domain of \( f^{-1} \).

\( x \neq 3 \)

(c) (2 pts) Evaluate \( f^{-1}(1) \)

\[
f^{-1}(1) = \frac{1 + 2}{1 - 3} = -\frac{3}{2}.
\]
5. (15 pts) Find the vertical asymptotes (you needn’t graph the functions):

(a) (5 pts) \( y = \sin x \)

None

(b) (5 pts) \( y = \frac{2x^2}{(x^2-4)(x-1)} \)

\( x = \pm 2, 1 \)

(c) (5 pts) \( y = \tan x \)

\( x = \frac{\pi}{2} + n\pi, \quad n = 0, \pm 1, \pm 2, \pm 3... \)
6. (20 pts) Differentiate: (Do not simplify.)

(a) (5 pts) \( y = x^5 + 6x^3 - 144 \)

\[ y' = 5x^4 + 18x^2 \]

(b) (5 pts) \( y = (2x^2 - 3)(x^{-2} + 1) \)

\[ y' = (2x^2 - 3)(-2x^{-3}) + (4x)(x^{-2} + 1) \]

(c) (5 pts) \( y = \frac{\cos x}{\sin x} \)

\[ y' = \frac{\sin x(-\sin x) - \cos x \cos x}{\sin^2 x} \]

(d) (5 pts) \( y = \cos (x^5 + 2x) \)

\[ y' = -(5x^4 + 2) \sin (x^5 + 2x) \]
7. (10 pts) Differentiate: \( f(x) = \frac{\sin^2(x + \sqrt{x})}{x \cos x} \)  \((\text{Do not simplify.})\)

\[
f'(x) = \frac{x \cos x [2 \sin (x + \sqrt{x})(1 - \frac{1}{2}x^{-1/2})] - \sin^2 (x + \sqrt{x})[\cos x - x \sin x]}{(x \cos x)^2}
\]
8. (12 pts) Let \( f(x) = x^2 + 5x \). Use the mathematical definition of derivative to derive a formula for \( f'(x) \).

(Recall the mathematical definition: \( f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} \).)

\[
\begin{align*}
f'(x) &= \lim_{h \to 0} \left\{ \frac{f(x+h) - f(x)}{h} \right\} \\
&= \lim_{h \to 0} \left\{ \frac{(x+h)^2 + 5(x+h) - x^2 - 5x}{h} \right\} \\
&= \lim_{h \to 0} \left\{ \frac{2xh + h^2 + 5x + 5h - 5x}{h} \right\} \\
&= \lim_{h \to 0} \left\{ 2x + h + 5 \right\} = 2x + 5.
\end{align*}
\]
9. (16 pts) Consider the following complicated equation:

\[ 2xy^3 + 4x^2 - 2y^4 = 0. \]

(a) (8 pts) Use implicit differentiation to obtain a formula for \( y' = \frac{dy}{dx} \) at a point \((x, y)\) on the graph.

\[ y' = -\frac{8x + 2y^3}{6xy^2 - 8y^3} \]

(b) (8 pts) Find the slope \( m \) and \( y \)-intercept \( b \) of the line tangent to the graph of \( f \) at the point \((1, -1)\).

\[ m = -\frac{3}{7}, \quad b = -\frac{4}{7} \]
10. (20 pts) Let \( y = f(x) = \frac{1}{4}x^4 - 2x^2 - 2 \).

(a) (5 pts) Find the values of \( x \) that are critical points.

\[
y' = x^3 - 4x = x(x^2 - 4) = 0,
\]
Critical points at \( x = 0, \pm 2 \).

(b) (5 pts) Use the second derivative test to determine the values of \( x \) that are relative maxima and relative minima.

\[
f''(x) = 3x^2 - 4,
\]
so

\[
f''(0) = -4 < 0, \quad f''(\pm 2) = (3)(4) - 4 > 0,
\]
implying \( x = 0 \) is a relative max and \( x = \pm 2 \) are relative min’s.

(c) (5 pts) Find the values of \( x \) that are inflection points.

\[
f''(x) = 3x^2 - 4 = 0 \text{ implies } x = \pm \frac{2}{\sqrt{3}} \text{ are the inflection points.}
\]

(d) (5 pts) Use interval notation (brackets) to describe the set of \( x \) where \( f \) is convex up.

\[
f''(x) = 3x^2 - 4 > 0 \text{ implies } x \in (-\infty, -\frac{2}{\sqrt{3}}) \cup (\frac{2}{\sqrt{3}}, +\infty).
\]
11. (12 pts) Two ships pass near a lighthouse. Let $x(t)$ denote the distance (in miles) from the first ship to the lighthouse, and $y(t)$ the distance from the second ship to the lighthouse, and assume you know that at every time, $x^2 + y^3 = 17$. Find the speed $\frac{dy}{dt}$ when the first ship is 3 miles from the lighthouse moving at speed $\frac{dx}{dt} = 1$ mile per hour.

$$x(t)^2 + y(t)^3 = 17$$

holds at every time, so

$$2xx' + 3y^2y' = 0$$

holds at every time. We know $x = 3$, $x' = 1$ and $x^2+y^3 = 17$ gives $y = 2$. Thus

$$y' = \frac{-6}{3 \cdot 4} = -\frac{1}{2}.$$
12. (12 pts) The cost of material to build a rectangular vertical wall $L$ feet long and $H$ feet high is $2H^2 + 3L$ dollars. Find the dimensions of the wall of area 36 square feet that minimizes the cost.

\[ C = 2H^2 + 3L, \quad \text{and} \quad L \cdot H = 36. \]

Solving $L = \frac{36}{H}$ and substituting gives

\[ C = 2H^2 + \frac{3 \cdot 36}{H}, \]

so

\[ \frac{dC}{dH} = 4H - \frac{3 \cdot 36}{H^2} = 0 \]

gives

\[ H^3 = 27, \]

or $H = 3$, $L = 12$. Range of variables $(0, \infty)$ with $C = \infty$ at the endpoints confirms this is the minimum.
13. (12 pts) The radius \( r \) of a circle is increasing at a rate of 5 meters per second.

(a) (6 pts) Find a formula for the rate \( \frac{dA}{dt} \) at which the area is increasing.

\[
A = \pi r^2, \quad \frac{dA}{dt} = 2\pi r \frac{dr}{dt} = 10\pi r
\]

(b) (6 pts) Use differentials to give the estimate \( dA \) for the increase in area over time interval \( dt = 2 \) seconds when the radius is 10 meters.

\[
dA = 10\pi r dr = 200\pi.
\]
14. (10 pts) Sketch the graph of a function continuous except at \( x = 2 \), such that
\[
\lim_{x \to 2^-} = -1, \quad \lim_{x \to 2^+} = +\infty, \quad f(2) = 1.
\]

Graph comes in from the left to an open dot at \((2, -1)\); there is a closed dot at \((2, 1)\); and the graph goes off to \( y = +\infty \) as it comes in to \( x = 2 \) from the right.
15. (12 pts) Find the point on the graph of \( y = \sqrt{x}, \ x \geq 0 \), closest to the point \((1,0)\).

\[
D = d^2 = (x - 1)^2 + (y - 0)^2, \quad y = \sqrt{x}
\]
gives

\[D = (x - 1)^2 + x.\]

Differentiating gives

\[
\frac{dD}{dx} = 2(x - 1) + 1 = 0,
\]
giving a unique critical point. Plugging in for \( y \) gives critical point on graph as \( x = \frac{1}{2}, \ y = \frac{1}{\sqrt{2}} \).

Now the distance \( D \to \infty \) as \( x \to +\infty \), and so the minimum distance must be at endpoint \( x = 0 \) or critical point \( x = \frac{1}{2} \). Checking \( D(0) = 1 \) and \( D(\frac{1}{2}) = \frac{1}{4} + \frac{1}{2} < 1, \) it follows that the closest point must is the critical point \( x = \frac{1}{2}, \ y = \frac{1}{\sqrt{2}} \).