FINAL EXAM
Math 16A
Temple-Fall 2012

−Print your name, section number and put your signature on the upper right-hand corner of this exam. Write only on the exam.

−Show all of your work, and justify your answers.

SCORES

#1

#2

#3

#4

#5

#6

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#8

#9

#10

TOTAL:

(1) (20 pts) Use the definition of derivative

$$\frac{dy}{dx} \equiv f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

to derive the formula for $f'(x)$ assuming $f(x) = 7x^2$.

Solution:

$$\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{7(x + \Delta x)^2 - 7x}{\Delta x} \\
&= 7 \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} \\
&= 7 \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} \\
&= 7 \lim_{\Delta x \to 0} (2x + \Delta x) \\
&= 7 \cdot 2x \\
&= 14x
\end{aligned} \tag{1}$$

(2) (20 pts) Differentiate: (Do not simplify.)

(a) $y = 2x^{10} + 7x^7 - 3$

$y' = 20x^9 + 49x^6$

(b) $y = (5x^2 + 1)\sin x$

$y' = 10x\sin x + (5x^2 + 1)\cos x$

(c) $y = \frac{\tan x}{\cos x}$

$y' = \frac{d}{dx}\frac{\sin x}{\cos^2 x} = \frac{\cos^2 x \cos x - \sin x(-2\cos x \sin x)}{\cos^4 x}$

(d) $y = \sin^3(3x)$

$y' = 9\sin^2(3x)$

(3) (28 pts) Assume $f(x) = (x-1)^2(x+1)$.

(a) Use the product rule to find $f'(x)$. Do not simplify.

Solution:

$$f'(x) = 2(x-1)(x+1) + (x-1)^2$$

(b) Factor $f'(x)$ and find all critical numbers.

Solution:

$$\begin{aligned} f'(x) &= (x-1)\{2(x+1) + (x-1)\} \\ &= (x-1)(3x+1)) \end{aligned}$$

Critical Numbers: $x = -1/3, 1$

(c) Apply the second derivative test to determine which critical numbers are relative max and relative min.

Solution:

$$f''(x) = (3x+1) + 3(x-1) = 6x - 2.$$

$f''(-1/3) = -4 < 0$, implies $x = -1/3$ is rel max;
$f''(1) = 4 > 0$, implies $x = 1$ is rel. min.

(d) Using the open bracket notation, determine the intervals on which f is increasing and decreasing.

Solution:

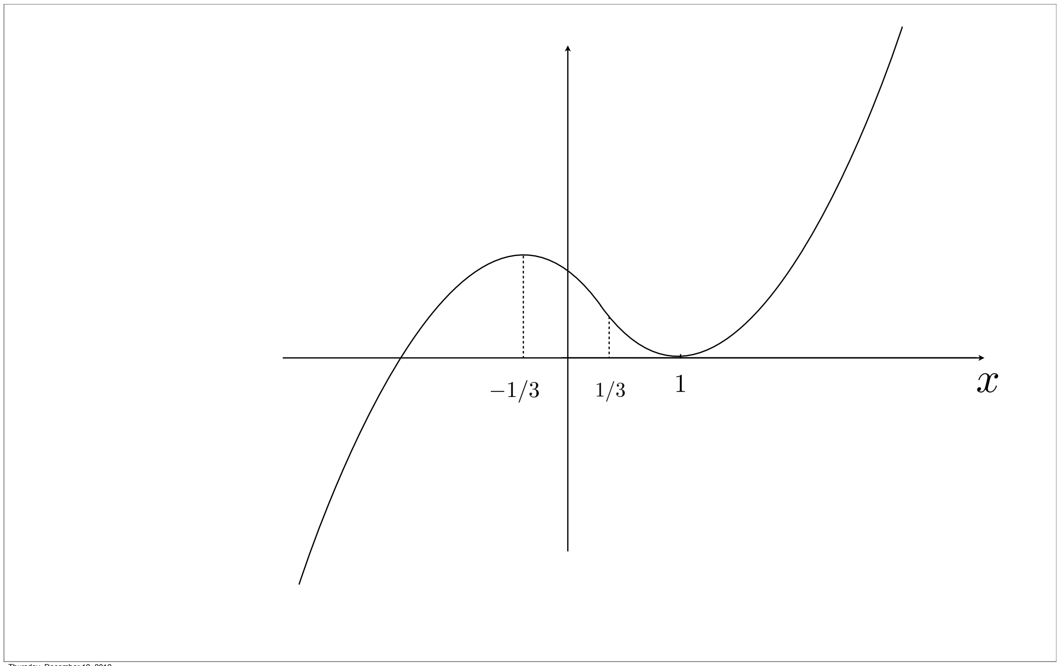
f *increasing* on $(-\infty, -1/3) \cup (1, +\infty)$ as $f'(x) > 0$;
f *decreasing* on $(-1/3, 1)$ as $f'(x) < 0$

(e) Determine the intervals in which f is concave up and concave down, and find all inflection points.

Solution: $f''(x) = 6x - 2$. So

f *concave down* on $(-\infty, 1/3)$ as $f''(x) < 0$;
f *concave up* on $(1/3, +\infty)$ as $f''(x) > 0$

(f) Graph f, indicating the above features.



Thursday, December 13, 2012

(g) Determine the absolute minimum of f on the interval $[-1, 1]$.

Solution: Minimum occurs at critical point or endpoint, so at $x = -1, 1$, or $-1/3$.

$f(-1) = (2^2)(0) = 0$;

$f(-1/3) = (-4/3)^2(2/3) = (16/9)(2/3) = 32/27 > 0$;

$f(1) = 0$.

Conclude: The minimum occurs at $x = -1, 1$.

(4) (15 pts) Find the vertical asymptotes (you needn't graph the functions):

(a) $y = \frac{1}{\sin x}$

Solution: $x = n\pi$, n a positive or negative integer.

(b) $f(x) = \frac{2x^2-8}{x^2-9}$

Solution: $x = -3, 3$

(c) $y = \tan x$

Solution: $x = \pi/2 + n\pi$, n a positive of negative integer.

(5) (20 pts) A ball thrown upward from initial height y_0 rises under the downward force of gravity according to the trajectory $y(t) = -16t^2 + v_0 t + y_0$, where v_0 is the initial upward velocity, y_0 is the initial height, and t is the time from launch in seconds. Derive a formula for the *time* T it takes the ball to reach its highest point.

Solution: $\frac{dy}{dt} = -32t + v_0 = 0$ implies $T = v_0/32$.

(6) (20 pts) A company's profit in millions of dollars for producing x jet airplanes is

$$P(x) = .005x^4 + 10x - 4.$$

Find the differential $dP = P'(x)dx$, and use it to find the marginal profit in producing two additional jets at production level $x = 10$.

Solution: $dP = [(.005)(4)x^3 + 10]dx$.

Since $x = 10$, $dP = [(.02)(10)^3 + 10]dx = 30dx$.

Since $dx = 2$, conclude that

$$dP = 30dx = (30)(2) = 60.$$

(7) (20 pts) (a) Use implicit differentiation to obtain a formula for $y' = \frac{dy}{dx}$ at a point (x, y) on the graph of

$$x^2y^3 + xy^2 - 2xy = 0.$$

Solution: Differentiate wrt x:

$2xy^3 + x^2 3y^2 y' + y^2 + 2xyy' - 2y - 2xy' = 0.$

Collecting y':

$(x^2 3y^2 + 2xyy - 2x)y' + 2xy^3 + y^2 - 2y) = 0.$

Solving for y':

$$y' = \frac{2xy^3 + y^2 - 2y}{3x^2y^2 + 2xy - 2x} = 0.$$

(b) Find the slope m and y-intercept b of the line tangent to the graph of f at the point $(1, 1)$.

Solution: $y'(1, 1)) = \frac{2+1-2}{3+2-2} = 1/3$. The point slope formula gives

$$(y - 1) = \frac{1}{3}(x - 1)$$

or

$$y = \frac{1}{3}x + \frac{2}{3}.$$

So $m = 1/3$ and $b = 2/3$.

(8) (17 pts) Let $g(x) = \sqrt{x}$, $h(x) = x - 1$.

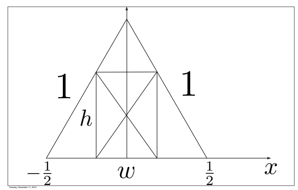
(a) Find the Domain of $f(x) = \frac{g(x)}{h(x)}$.

Solution: $f(x) = \frac{g(x)}{h(x)} = \frac{\sqrt{x}}{x-1}$.
So the Domain is $x \neq 1$ and $x \geq 0$. In set notation this would be $[0, 1) \cup (1, \infty)$.

(b) Find the Domain of $f(x) = (g \circ h)(x) = g(h(x))$.

Solution: $f(x) = (g \circ h)(x) = g(h(x)) = \sqrt{x - 1}$.
So the Domain is $x - 1 \geq 0$, or in set notation this would be $[1, +\infty)$.

(9) (20 pts) A beam of width w and height h is cut from an equilateral triangle of side $a = 1ft$. Find the dimensions of the beam that maximizes the strength $S = wh^2$.



Solution: The center height of the triangle is $\sqrt{3}/2$. Thus the line h vs x in the picture has slope $m = -\frac{\frac{\sqrt{3}}{2}}{1/2} = \sqrt{3}$ and y intercept $\sqrt{3}/2$. That is,

$$h = -\sqrt{3}\,x + \frac{\sqrt{3}}{2}.$$

It follows that when $x = w/2$,

$$h = -\sqrt{3}x + \frac{\sqrt{3}}{2} = -\sqrt{3}\frac{w}{2} + \frac{\sqrt{3}}{2}.$$

The strength we wish to maximize is thus $S = wh^2$ so to make it simple, solve for w and substitute:

$$w = -\frac{2}{\sqrt{3}}h + 1,$$

$$S = (-\frac{2}{\sqrt{3}}h + 1)h^2 = (-\frac{2}{\sqrt{3}})h^3 + h^2.$$

Taking the derivative to find the critical numbers,

$$S'(h) = 3\left(-\frac{2}{\sqrt{3}}\right)h^2 + 2h = h\left(3(-\frac{2}{\sqrt{3}})h + 2\right) = 0,$$

or

$$h = \frac{2}{3\frac{2}{\sqrt{3}}} = \frac{\sqrt{3}}{3}, \quad w = 1 - \frac{2}{\sqrt{3}}\frac{\sqrt{3}}{3} = 1/3.$$

(10) (20 pts) A 10 foot ladder is leaning against a house. The base of the ladder is pulled away from the house at a rate of 1 foot per second. How fast is the top of the ladder moving down the wall when the base is 6 feet from the house.

Solution: Label the vertical height to the top of the ladder y and the distance from the bottom of the ladder x. Then

$$x^2 + y^2 = 10.$$

Assuming $x = x(t),\ y = y(t)$, differentiate implicitly:

$$2x\dot{x} + 2y\dot{y} = 0.$$

Solve for $\dot{y}$:

$$\dot{y} = -\frac{x\dot{x}}{y}.$$

At the point $x = 6$, we know $6^2 + 8^2 = 10^2$ so $y = 8$. I follows that

$$\dot{y} = -\frac{6 \cdot 1}{8} = -\frac{3}{4} \text{ ft/second.}$$