MIDTERM EXAM I
Math 16A
Temple-Winter 2010

−Print your name, section number and put your signature on the upper right-hand corner of this exam. Write only on the exam.

−Show all of your work, and justify your answers for full credit.

<u>SCORES</u>

#1

#2

#3

#4

#5

#6

#7

#8

TOTAL:

1. Determine the following limits (simplify answer):

(a) (6 pts) $\lim_{x\to 2}\sqrt{\frac{x^2-4}{x-2}}$ (Hint: Factor)

$$\begin{aligned}\lim_{x\to 2}\sqrt{\frac{x^2-4}{x-2}} &= \lim_{x\to 2}\sqrt{\frac{(x-2)(x+2)}{x-2}}\\ &= \lim_{x\to 2}\sqrt{\frac{(x+2)}{1}} = \sqrt{4} = 2\end{aligned}$$

(b) (6 pts) $\lim_{x\to 0}\frac{\sqrt{x+2}-\sqrt{2}}{4x}$ (Hint: Conjugate)

$$\begin{aligned}\lim_{x\to 0}\frac{\sqrt{x+2}-\sqrt{2}}{4x} &= \lim_{x\to 0}\left\{\frac{\sqrt{x+2}-\sqrt{2}}{4x}\cdot\left(\frac{\sqrt{x+2}+\sqrt{2}}{\sqrt{x+2}+\sqrt{2}}\right)\right\}\\ &= \lim_{x\to 0}\left\{\frac{x+2-2}{4x(\sqrt{x+2}+\sqrt{2})}\right\}\\ &= \lim_{x\to 0}\left\{\frac{1}{4(\sqrt{x+2}+\sqrt{2})}\right\}\\ &= \frac{1}{8\sqrt{2}} = \frac{\sqrt{2}}{16}\end{aligned}$$

(No simplification required past $\frac{1}{8\sqrt{2}}$.)

2. Determine the following limits:

(c) (6 pts) $\lim_{x\to\infty} \frac{2x^5+3x-4}{3x^5-7}$ (Hint: Divide by highest power)

$$
\begin{aligned}
\lim_{x\to+\infty} \frac{2x^5+3x-4}{3x^5-7} &= \lim_{x\to+\infty} \left\{ \frac{2x^5+3x-4}{3x^5-7} \left(\frac{1/x^5}{1/x^5} \right) \right\} \\
&= \lim_{x\to+\infty} \left\{ \frac{2+\frac{3}{x^4}-\frac{4}{x^5}}{3-\frac{7}{x^5}} \right\} \\
&= \frac{2}{3}
\end{aligned}
$$

(d) (6 pts) $\lim_{x\to 0} \frac{1-\cos^2 x}{\tan^2 x \cos x}$ (Hint: Simplify)

$$
\begin{aligned}
\lim_{x\to 0} \frac{1-\cos^2 x}{\tan^2 x \cos x} &= \lim_{x\to 0} \left\{ \frac{\sin^2 x}{\frac{\sin^2 x}{\cos^2 x} \cos x} \right\} \\
&= \lim_{x\to 0} \{\cos x\} \\
&= 1
\end{aligned}
$$

3. Let $f(x) = \sqrt{4 - x^2}, \qquad g(x) = 2\sin x.$

(a) (5 pts) Find the Domain of f.

$$\text{Domain}(f) = \{x : \quad -2 \leq x \leq 2\}.$$

(b) (5 pts) Find the (precise) Range of g.

$$\text{Range}(g) = \{x : \quad -2 \leq x \leq 2\}.$$

(c) (6 pts) Find $(f \circ g)(x)$.

$$(f \circ g)(x) = \sqrt{4 - 4\sin^2 x}.$$

(d) (5 pts) Prove that $(f \circ g)(x)$ is defined for every real number x. (I.e, show the Domain of $f \circ g$ is all of $\mathcal{R}$.)

- Since the domain of g is all real numbers and the range of g lies within the domain of f, it follows that the domain of $f \circ g = f(g(x))$ equals the domain of g equals all of $\mathcal{R}$.

- Or...Let $x \in \mathcal{R}$. Then $-2 \leq g(x) \leq 2$, so $4 - g(x)^2 \geq 0$, and thus $f(g(x)) = \sqrt{4 - g(x)^2}$ is well defined.

4. Consider the function $f(x) = \frac{2x+1}{x-3}$ with Domain $x \neq 3$.

(a) (7 pts) Find a formula for $f^{-1}(x)$.

Set $y = \frac{2x+1}{x-3}$, solve for x and then switch x and y:

$$\begin{aligned} xy - 3y &= 2x + 1 \\ x(y-2) &= 3y + 1 \\ x &= \frac{3y+1}{y-2} \end{aligned}$$

So

$$f^{-1}(x) = \frac{3x+1}{x-2}$$

(b) (4 pts) Find the Domain of f^{-1}.

$$\text{Domain}(f^{-1}) = \{x \in \mathcal{R} : \ x \neq 2\}$$

(c) (4 pts) Evaluate $f^{-1}(1)$

$$f^{-1}(1) = \frac{3 \cdot 1 + 1}{1 - 2} = -4$$

5. Find all vertical and horizontal asymptotes (you needn't graph the functions):

   (a) (3 pts) $y = \sin x$

   No vertical or horizontal asymptotoes

   (b) (6 pts) $y = \frac{2x^2}{(x^2-1)(x+2)}$

   Vertical asymptotes at $x = -1, 1, -2$; Horizontal asymptote $y = 0$.

   (c) (6 pts) $y = \tan x$

   Since $y = \tan x = \frac{\sin x}{\cos x}$, vertical asymptotes are the zeros of $\cos x$ which are the lines $x = \frac{\pi}{2} + n\pi$ for $n \in \mathcal{N}$. No limits at $x \to \pm\infty$, so no horizontal asymptotes.

6. Given a function $f(x)$:

(a) (4 pts) State the *definition* of the derivative $f'(x)$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

(b) (7 pts) Directly from the definition, derive the value of $f'(2)$ if $f(x) = x^3$. (Hint: $(x-a)^3 = x^3 - 3ax^2 + 3a^2x - a^3$).

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \left\{ \frac{(x+h)^3 - x^3}{h} = \frac{x^3 + 3hx^2 + 3h^2x + h^3 - x^3}{h} \right. \\ &= \left. \frac{3hx^2 + 3h^2x + h^3}{h} = 3x^2 + 3hx - h^2 \right\} = 3x^2 \end{aligned}$$

7. (7 pts) Draw the graph of a function continuous except at $x = 2$, such that $f(2) = 1$, $\lim_{x \to 2-} = 0$, and $\lim_{x \to 2+} = 4$. (Use closed and open dots correctly at $x = 2$.)

8. (7 pts) Find the equation of the line *perpendicular* to the graph of $y = 2x^2 - 3$ at the point $(1, -1)$.

Calculate $f'(x) = 4x$, so $f'(1) = 4$, thus the perpendicular slope is $m = -\frac{1}{4}$, and the line is

$$\frac{y+1}{x-1} = -\frac{1}{4}$$

or

$$y = -\frac{1}{4}x - \frac{3}{4}$$