MIDTERM EXAM II—SOLUTIONS
Math 16A
Temple-Fall 2012

−Print your name and put your signature on the upper right-hand corner of this exam. Write only on the exam.

−Show all of your work, and justify your answers for full credit.

<u>SCORES</u>

#1

#2

#3

#4

#5

#6

#7

TOTAL:

1. Differentiate: (Do not simplify.)

(a) (4 pts) $y = \frac{1270}{730} - 2x^{21} + 4x^{11}$

$f'(x) = -42x^{20} + 44x^{10}$

(b) (4 pts) $f(x) = \{\tan(x)\}\{\sin x\}$

$f'(x) = \sec^2 x \ \sin x + \tan x \ \cos x$

(c) (4 pts) $y = \frac{2x^3-3}{3x^2+1}$

$f'(x) = \frac{(3x^2+1)(6x^2)-(2x^3-3)(6x)}{(3x^2+1)^2}$

(d) (4 pts) f(x)=sin (x^4+1)

$f'(x) = \cos(x^4+1)\, 4x^3$

2. (15 pts) Differentiate: $f(x) = \frac{\sin^3(x+\sqrt{x})}{x \tan x}$ (Do not simplify.)

$$f'(x) = \frac{(x\tan x)(3\sin^2(x+\sqrt{x})\,(1+\frac{1}{2\sqrt{x}}) - (\sin^3(x+\sqrt{x}))(\tan x + x\sec^2 x)}{(x\tan x)^2}$$

3. Assume the height y in feet of a falling object after t seconds is given by

$$y = -16t^2 + 32t + 48.$$

(a) (4 pts) Find the velocity $v = \frac{dy}{dt}$ as a function of t.

$$v = \frac{dy}{dt} = -32t + 32 \quad \frac{\text{ft}}{\text{s}}$$

(b) (4 pts) Find the acceleration $a = \frac{dv}{dt}$ as a function of t.

$$a = \frac{dv}{dt} = -32 \quad \frac{\text{ft}}{\text{s}^2}$$

(c) (4 pts) Find the velocity at $t = 0$.

$$v_0 = 32 \quad \frac{\text{ft}}{\text{s}}$$

(d) (4 pts) Find the highest the object goes. (Hint: $v = \frac{dy}{dt} = 0$ at the moment when the highest point is reached.) Setting $v = 0$ in $v = -32t + 32$ gives $32t = 32$ or $t = 1$. The highest point happens at $t = 1$ at which point the height is given by

$$y = -16(1)^2 + 32(1) + 48 = 16 + 48 = 64 \text{ feet.}$$

4. The cost of making x jet airplanes in millions of dollars is

$$C(x) = 4x - \sqrt{x}.$$

Recall that the marginal cost of producing dx more airplanes than x is dy where

$$\frac{dy}{dx} = C'(x).$$

(a) (6 pts) Find the cost of producing 100 airplanes.

$$C(100) = 4 \cdot 100 - \sqrt{100} = 400 - 10 = 390 \text{ million dollars.}$$

(b) (8 pts) Find the marginal cost of producing two more airplanes if $x = 100$. (Hint: Solve for dy.)

$$C'(x) = 4 - \frac{1}{2\sqrt{x}},$$

so

$$C'(100) = 4 - \frac{1}{2\sqrt{100}} = 4 - \frac{1}{20}.$$

Thus

$$dy = \left(4 - \frac{1}{20}\right) \cdot 2 = 8 - \frac{1}{10} = 7\frac{9}{10} \text{ million dollars.}$$

5. (13 pts) Using the definition of derivative,

$$\frac{d}{dx}f(x) \equiv f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x},$$

give a careful proof that $\frac{d}{dx}x^2 = 2x$.

$$\begin{aligned}
\frac{d}{dx}x^2 &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - (x)^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(x^2 + 2x\Delta x + (\Delta x)^2) - x^2)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} (2x + \Delta x)) = 2x.
\end{aligned}$$

6. (13 pts) Using the definition of derivative,

$$\frac{d}{dx} f(x) \equiv f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x},$$

verify the product rule:

$$\frac{d}{dx} \{f(x) g(x)\} = f'(x) g(x) + f(x) g'(x).$$

Solution:

$$\frac{d}{dx} (f(x) + g(x)) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)+g(x+\Delta x)-f(x)-g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left\{ \frac{f(x+\Delta x)-f(x)}{\Delta x} g(x + \Delta x) + f(x) \frac{g(x+\Delta x)-g(x)}{\Delta x} \right\}$$

$$= \lim_{\Delta x \to 0} \left\{ \frac{f(x+\Delta x)-f(x)}{\Delta x} g(x + \Delta x) \right\} + \lim_{\Delta x \to 0} \left\{ f(x) \frac{g(x+\Delta x)-g(x)}{\Delta x} \right\}$$

$$= f'(x) g(x) + f(x) g'(x)$$

7. (13 pts) Using the definition of derivative,

$$\frac{d}{dx}f(x) \equiv f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x},$$

give a careful proof that $\frac{d}{dx}\sin(x) = \cos(x)$. You may use the fact that

$$\lim_{\Delta x \to 0} \frac{\sin(\Delta x)}{\Delta x} = 1,$$

and

$$\lim_{\Delta x \to 0} \frac{1 - \cos(\Delta x)}{\Delta x} = 0.$$

Solution:

$$\frac{d}{dx}\sin x = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x)-\sin(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sin(x)\cos\Delta x+\sin(\Delta x)\cos x-\sin(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left[\left\{\sin x\frac{\cos(\Delta x)-1}{\Delta x}\right\} + \left\{\cos x\frac{\sin(\Delta x)}{\Delta x}\right\}\right]$$

$$= \lim_{\Delta x \to 0} \left\{\sin x\frac{\cos(\Delta x)-1}{\Delta x}\right\} + \lim_{\Delta x \to 0} \left\{\cos x\frac{\sin(\Delta x)}{\Delta x}\right\}$$

$$= \sin x \lim_{\Delta x \to 0} \left\{-\frac{1-\cos(\Delta x)}{\Delta x}\right\} + \cos x \lim_{\Delta x \to 0} \left\{\frac{\sin(\Delta x)}{\Delta x}\right\}$$

$$= \cos x$$