MIDTERM EXAM II–SOLUTIONS
Math 16A
Temple-Winter 2010

−Print your name, section number and put your signature on the upper right-hand corner of this exam. Write only on the exam.

−Show all of your work, and justify your answers for full credit.

SCORES

#1

#2

#3

#4

#5

#6

#7

TOTAL:
1. Differentiate: (Do not simplify.)

(a) (4 pts) \[ y = -x^{13} + 6x^2 - \frac{127}{73} \] (Hint: Power Rule)

\[ y' = -13x^{12} + 12x \]

(b) (4 pts) \[ f(x) = \{\sin(x)\} \{\cos x\} \] (Hint: Product Rule)

**Note: \( f(x) = \{\sin(x)\} \{\cos (x)\} \) was corrected on board during exam***********

\[ y' = (\cos x)(\cos x) + (\sin x)(-\sin x) = \cos^2 x - \sin^2 x \]

(c) (4 pts) \[ y = \frac{2x^2 - 3}{x^3 + 1} \] (Hint: Quotient Rule)

\[ y' = \frac{(x^3 + 1)(4x) - (2x^2 - 3)(3x^2)}{(x^3 + 1)^2} \]

(d) (4 pts) \[ f(x) = \sin(2x^2 + 1) \] (Hint: Chain Rule)

\[ y' = \cos(2x^2 + 1) \cdot 4x \]
2. (10 pts) Differentiate: \( f(x) = \frac{\tan^2(x + \sqrt{x})}{x \sin x} \) (Do not simplify.)

\[
f'(x) = \frac{(x \sin x)(2 \tan (x + \sqrt{x}) \sec^2 (x + \sqrt{x})(1 + \frac{1}{2}x^{-1/2}) - (\tan^2 (x + \sqrt{x}))(\sin x + x \cos x))}{(x \sin x)^2}
\]
3. (10 pts) A population of bacteria is introduced into a culture. The number of bacteria $P$ can be modeled by

$$P = 450 \left(1 + \frac{t}{5 + t^2}\right),$$

where $t$ is in hours. Find the rate of population growth after one hour.

Solution: Rate $= \frac{dP}{dt} = 450 \left(\frac{(5+t^2)-2t^2}{(5+t^2)^2}\right)$. At $t = 1$ the rate is:

$$P'(1) = 450 \cdot \frac{6 - 2}{36} = 450 \cdot \frac{1}{9} = 50 \text{ per hour.}$$
4. Let $y = f(x)$ where $f(x) = \frac{1}{3}x^3 - x^2 - 3x + 1$.

(a) (3 pts) Find $f'(x)$.
Solution: $f'(x) = x^2 - 2x - 3 = (x - 3)(x + 1)$.

(b) (3 pts) Use brackets to describe where $f$ is increasing
Solution: $f'(x) > 0$ when $x \in (-\infty, -1) \cup (3, \infty)$.

(c) (3 pts) Use brackets to describe where $f$ is decreasing
Solution: $f'(x) < 0$ when $x \in (-1, 3)$.

(d) (3 pts) Find $(x, y)$ where $f$ takes a (local) maximum.
Solution: $x = -1, \quad y = -\frac{1}{3} - 1 + 3 + 1 = 2\frac{2}{3}$.

(e) (3 pts) Find $(x, y)$ where $f$ takes a (local) minimum.
Solution: $x = 3, \quad y = 9 - 9 - 9 + 1 = -8$.

(f) (3 pts) Sketch the graph of $f$ labeling (b)-(e).
Solution: “Up from $-\infty$ to $2\frac{2}{3}$ at $x = -1$, back down to $-8$ at $x = 3$, back up to $\infty$.”
5. (15 pts) A ball is thrown upward and forward. Its vertical height (in feet) after \( t \) seconds is given by

\[
y(t) = -16t^2 + 8t.
\]

**Note:** Misprint on exam \( y(t) = 16t^2 - 8t \) was corrected on board during exam

Before hitting the ground it moves horizontally forward at constant velocity \( \frac{dx}{dt} = 4 \) feet per second.

(a) (5 pts) Find a formula for the vertical velocity \( \frac{dy}{dt} \).

Solution: \( v = \frac{dy}{dt} = -32t + 8 \).

(b) (5 pts) Find a formula for the vertical acceleration.

Solution: \( a = -32 \) ft/sec\(^2\)

(c) (5 pts) Find the highest the ball goes up.

Solution: \(-32t + 8 = 0\) so \( t = \frac{8}{32} = 1/4 \) sec., so \( y_{max} = -16 \left( \frac{1}{4^2} \right) + 8 \frac{1}{4} = -1 + 2 = 1 \) ft.

(d) (5 pts) Find the time at which the ball hits the ground.

Solution: \( y = -16t^2 + 8t = 8t(-2t + 1) = 0\) so \( t = 0, 1/2 \).

(e) (5 pts) Find how far the ball traveled horizontally forward when it hit the ground.

Solution: Horizontal velocity = 4 feet per second, it travels for 1/2 second, so \( d = (1/2)(4) = 2 \) ft.
6. Consider the following complicated equation:

\[ x^2y^3 - 2xy - 1 = 0. \]  

(1)

(a) (8 pts) Obtain a formula for \( \frac{dy}{dx} \) the slope of the line tangent to the graph of (1) at a point \((x, y)\) on the graph.

Solution: (Implicit Differentiation)

\[
2xy^3 + 3x^2y^2 \frac{dy}{dx} - 2y - 2x \frac{dy}{dx} = 0
\]

\[
\{3x^2y^2 - 2x\} \frac{dy}{dx} + 2xy^3 - 2y = 0.
\]

Solving for \( \frac{dy}{dx} \) gives:

\[
\frac{dy}{dx} = \frac{-2xy^3 + 2y}{3x^2y^2 - 2x}
\]

(b) (8 pts) Find the equation of the line tangent to the graph of (1) at the point \((1, -1)\).

Solution: At \((x, y) = (1, -1)\),

\[
\frac{dy}{dx} = \frac{2(1)(-1)^3 - 2(-1)}{2 - 3(1)^2(-1)^2} = 0.
\]

So the equation of the line is:

\[
\frac{y - (-1)}{x - 1} = 0,
\]

or

\[
y = -1.
\]
7. (15 pts) A car is moving parallel and along the shore of a straight river three miles wide. A policeman on the shore across the river from the car uses radar to determine that at the moment the car is 5 miles from him, it is receding from him at a velocity of 120 miles an hour. Draw a correct picture, label the relevant variables, and determine the velocity of the car at the moment it is 5 miles away from the policeman. (Hint: draw a picture and let $x(t)$ denote the diagonal distance from the policeman to the car, and let $y(t)$ denote the distance from the car to the point directly across the river from the policeman.)

**Note:** The clarification *The rate at which the distance between the car and the policeman was increasing was 120mph* was written on the board during exam**************

Solution: $y(t)^2 + 3^2 = x(t)^2$ so $2yy' = 2xx'$, and so

$$y' = \frac{xx'}{y} = \frac{(5)(120)}{4} = 150 \text{ mph}$$