MIDTERM EXAM III
Math 16A
Temple-Fall 2012

−Print your name and put your signature on the upper right-hand corner of this exam. Write only on the exam.

−Show all of your work, and justify your answers for full credit.

SCORES

#1

#2

#3

#4

#5

TOTAL:

1. Consider the following function: $f(x) = -x^3 - x^2 + x - 1$

(a) (5 pts) Find all critical numbers.

Soln:

$$f'(x) = -3x^2 - 2x + 1 = (3x - 1)(-x - 1) = 0$$

so critical points are $x = 1/3$ and $x = -1$.

(b) (5 pts) Apply the second derivative test to determine which critical numbers are relative max and relative min.

Soln:

$$f''(x) = -6x - 2$$

Since $f''(1/3) = -6(1/3) - 2 = -4 < 0$, $x = 1/3$ is a relative max.

Since $f''(-1) = -6(-1) - 2 = 4 > 0$, $x = -1$ is a relative min.

(c) (5 pts) Using the open bracket notation, determine the intervals on which f is increasing and decreasing.

Soln: Increasing where $f'(x) > 0$ which is $x \in (-1, 1/3)$, and decreasing where $f'(x) < 0$ which is $x \in (-\infty, -1) \cup (1/3, \infty)$.

(d) (5 pts) Determine the intervals in which f is concave up and concave down, and find all inflection points.

Soln: Concave up where $f''(x) > 0$ or $x \in (-\infty, -1/3)$.

Concave down where $f''(x) < 0$ or $x \in (-1/3, \infty)$.

Inflection point $x = -1/3$.

(e) (5 pts) Graph f, indicating the above features.

Soln: And S-shaped cubic going to $+\infty$ as $x \to -\infty$ and to $-\infty$ as $x \to +\infty$. Critical points are at the transitions from increasing to decreasing, inflection from concave up to concave down.

(e) (5 pts) Determine the absolute minimum of f on the interval $[-2, 0]$.

Soln: The absolute min occurs at either a critical point or at the endpoint $x = -2$ or $x = 0$. The only critical point in $(-2, 0)$ is $x = -1$. Checking the three points gives:

$$f(-1) = -(-1)^3 - (-1)^2 + (-1) - 1 = -2.$$

$$f(0) = -(0)^3 - (0)^2 + (0) - 1 = -1.$$

$$f(-2) = -(-2)^3 - (-2)^2 + (-2) + 1 = 8 - 4 - 2 - 1 = 1.$$

Conclude that $x = -1$ is where the minimum occurs, and the minimum is $f(-1) = -2$.

2. (10 pts) Determine the vertical and horizontal asymptotes of the function

$$f(x) = \frac{x^2 - 9}{x^2 - 3x + 2}$$

Soln: Factoring numerator and denominator gives

$$f(x) = \frac{(x-3)(x+3)}{(x-2)(x-1)},$$

so the vertical asymptotes are $x = 1$ and $x = 2$, and the horizontal asymptotes are given by

$$y = \lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{1 - 9/x^2}{1 - 3/x + 2/x^2} = 1.$$

3. (20 pts) Consider the curve described by the relation

$$x^2y^3 + \sin(y+1) + 1 = 0.$$

Verify $(1,-1)$ lies on the curve, and use implicit differentiation to find the slope of the graph of $f(x)$ at $(1,-1)$. Use this to find the point-slope form of the equation of the line tangent to the graph of f at the point $(-1,1)$.

Soln: Checking...$(1)^2(-1)^3+\sin(-(1)+1)+1 = -1+1 = 0$, which verifies that $(1,-1)$ is on the graph. Differentiating both sides implicitly treating $y = y(x)$ gives:

$$2xy^3 + x^2 2y^2 y' + \cos(y+1)y' = 0,$$

which at the point $x = 1,\ y = -1$ gives

$$2(1)(-1)^3 + (1)^2 2(-1)^2 y' + \cos(-1+1)y' = 0,$$

or

$$-2 + y' + 1 = 0,$$

which give slope $y' = m = 1$ at the point $(1,-1)$. The equation for the line tangent to the graph at $(1,-1)$ is thus

$$y + 1 = 1(x-1)$$

or

$$y = x - 2.$$

4. (20 pts) A ball thrown upward from initial height y_0 rises under the downward the force of gravity according to the trajectory $y(t) = -16t^2 + v_0 t + y_0$, where v_0 is the initial upward velocity. Derive a formula for how high the ball rises.

Soln: Maximum height is achieved when

$$\frac{dy}{dt} = -32t + v_0 = 0,$$

which happens when

$$t = \frac{v_0}{32}.$$

Putting this time in the formula for $y(t)$ gives

$$y_{max} = y\left(\frac{v_0}{32}\right) = -16\left(\frac{v_0}{32}\right)^2 + v_0\left(\frac{v_0}{32}\right) + y_0 = \frac{v_0^2}{64} + y_0,$$

which is the formula for the maximum height.

5. (20 pts) An airplane is crashing to the ground along a vertical straight line that is 3000 meters from a radar station (measured horizontally). The radar station measures the rate of change of distance between the airplane and the station is decreasing at 50 meters per second when the plane is at an altitude of 4000 meters. How fast is the plane losing altitude at that moment? (Hint: Draw a picture.)

Soln: Let $d(t)$ denote the distance from the radar station to the plane, and let $x(t)$ denote the height of the plane. Then the Pythagorean Theorem gives

$$d(t)^2 = (3000)^2 + x(t)^2.$$

Differentiating gives:

$$2d\dot{d} = 2x\dot{x},$$

and so solving for $\dot{x}$ gives

$$\dot{x} = \frac{d}{x}\dot{d}.$$

Now $x = 4000$, $d = 5000$ at the critical time, so putting these together with $\dot{d} = -50$ into this formula gives the rate of change of vertical height as

$$\dot{x} = \frac{5000}{4000}(-50) = -\frac{5}{4}50 = -\frac{5}{2}25 = -\frac{125}{2}m/s.$$