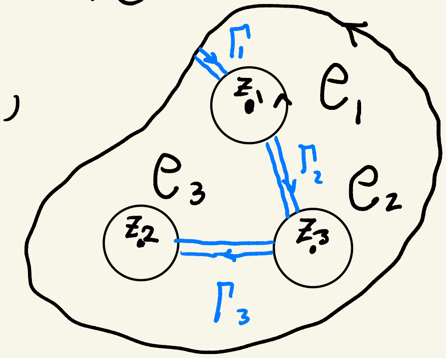
VIII Residue Theorem:

- **Main Idea:** Assume f is analytic at all but a finite number of points $z_1, \dots, z_n$. For example, $f(z) = \frac{p(z)}{q(z)}$, where p, q are polynomials, is analytic (by quotient rule) everywhere except at the zeroes of q. ($f(z) = \frac{p(z)}{q(z)}$ is called rational function)

- **Let** C be a scc which encloses $z_1, \dots, z_n$, so $z_1, \dots, z_n$ are inside C, and assume f is analytic in a nbhd of C, except at $z_1, \dots, z_n$.



Let $C_1, \dots, C_n$ be scc's such that only z_i is inside C_i.
Wlog, we can take C_i to be positively oriented circles with radius ε_i and center C_i, I.e.,

$$C_i = \partial B_{\varepsilon_i}(z_i), \quad i = 1, \dots, n.$$

- **Now** use our trick that $\oint_{\bar{C}} f(z)\,dz = 0$ if f is analytic inside $\bar{C}$, so **take**

$$\bar{C} = C + \Gamma_1 - C_1 - \Gamma_1 + \Gamma_2 - C_2 - \Gamma_2 + \cdots + \Gamma_n - C_n - \Gamma_n$$

(I.e., traverse the C_i and Γ_i so inside is on LEFT)

Thus the Γ_i-integrals cancel and we get

$$0 = \int_{\bar{e}} f(z)dz = \int_{e} f(z)dz - \int_{e_1} f(z)dz \cdots - \int_{e_n} f(z)dz$$

So

$$\int_{e} f(z)\,dz = \int_{e_1} f(z)dz + \cdots + \int_{e_n} f(z)dz = \sum_{i=1}^{n} \int_{e_i} f(z)dz$$

Conclude: To evaluate $\int_e f(z)dz$ it suffices to evaluate each $\int_{e_i} f(z)dz$ separately.

- Now assume we can expand f in powers of $(z-z_i)$ in an annulus $0<|z-z_i|<R_i$ around each singularity z_i of f: (wlog $\varepsilon_i < R_i$)

$$f(z) = \underbrace{\sum_{k=-\infty}^{\infty} c_k (z-z_i)^k}_{\text{Laurent expansion of } f \text{ about } z=z_i} = \underbrace{\sum_{k=1}^{\infty} \frac{c_{-k}}{(z-z_i)^k}}_{\text{negative powers}} + \underbrace{\sum_{k=0}^{\infty} c_k (z-z_i)^k}_{\text{positive "Taylor Series terms"}}$$

Now recall

$$\oint_{e_i} c_k (z-z_i)^k \, dz = \begin{cases} 0 & k \neq -1 \\ 2\pi i & k=-1 \end{cases}$$

I.e. only $\oint_{e_i} \frac{dz}{z-z_i} = 2\pi i \neq 0$ because all other powers have anti-derivative!

Conclude: If we can integrate the Laurent Series Term by Term (TxT), then we can evaluate $\oint_{C_i} f(z)\,dz$ as follows:

$$\oint_{C_i} f(z)\,dz = \oint_{C_i} \sum_{k=-\infty}^{\infty} C_k (z-z_i)^k\,dz \underset{\text{Needs Justification}}{=} \sum_{k=-\infty}^{\infty} \underbrace{\oint_{C_i} C_k (z-z_i)^k\,dz}_{=0\ \forall\, k\neq -1}$$

$$= C_{-1} \oint_{C_i} \frac{dz}{z-z_i}$$

$$= 2\pi i\, C_{-1}$$

Defn: C_{-1} = Coeff of $\frac{1}{z-z_i}$ in the Larent Expansion of f about $z=z_i$ = the residue of f at z_i

$$C_{-1} = R(f; z_i) \qquad \text{"residue of } f \text{ at } z_i\text{"}$$

Theorem (Residue Theorem)

$$\int_C f(z)\,dz = 2\pi i \sum_{i=1}^{n} R(f; z_i)$$

Conclude: The problem of evaluating integrals reduces to the problem of finding residues.

Residue Theorem: If f is analytic in a nbhd of a scc C except at a finite number of point singularities $z_1, \dots, z_n$, then f has a unique Laurent expansion about each z_i which converges in the annulus

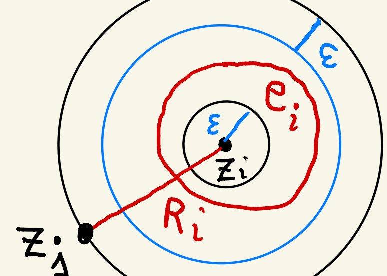
$$0 < |z - z_i| < R_i$$

where R_i is the distance from z_i to the nearest singularity in f – and converges uniformly and absolutely in every sub-annulus. If C is in a sub-annulus then $\varepsilon \le |z - z_i| \le R_i - \varepsilon$.

$$\oint_{C_i} f(z)\,dz = 2\pi i\, R(f; z_i)$$

where $R(f; z_i) = C_{-1}^{i}$ = coeff of $\frac{1}{z - z_i}$ in the Laurent expansion of f at z_i.

To make this precise – we need to prove the Laurent expansion converges in $0 < |z - z_i| < R_i$, absolutely and uniformly in every sub-annulus $\varepsilon < |z - z_i| < R - \varepsilon$.



- Uniform convergence - what we need to pass the integral sign thru the summation sign.

The following general theorem is all we need:

Q: When does $\sum_{n=1}^{\infty} g_n(z)$ converge uniformly and absolutely $\forall\, z \in D_{\text{open}}$?

Theorem (Weierstrass M-test) Assume

(1) $|g_n(z)| \le M_n \quad \forall\, z \in D, \; n = 1, 2, 3 \ldots$

(2) $\sum_{n=1}^{\infty} M_n = \lim_{N \to \infty} \underbrace{S_N}_{\text{partial sum}} = L < \infty, \quad S_N = \sum_{n=1}^{N} M_n$

Then $\sum_{n=1}^{\infty} g_n(z)$ converges to a limit function $f(z)$ absolutely and uniformly. I.e.

$\lim_{N \to \infty} \sum_{n=1}^{N} g_n(z) = f(z) \in \mathbb{C} \quad \forall z \in D.$ (converges)

$\lim_{N \to \infty} \sum_{n=1}^{N} |g_n(z)| < \infty \quad \forall z \in D.$ (Converges absolutely)

$\forall \varepsilon > 0 \; \exists M \text{ st } N > M \Rightarrow \left| \sum_{n=1}^{N} g_n(z) - f(z) \right| < \varepsilon \quad \forall z \in D.$

(converges uniformly)

Defn: $\sum_{n=1}^{\infty} g_n(z)$ converges absolutely if
$$\sum_{n=1}^{\infty} |g_n(z)| < \infty,$$
i.e., converges $\forall z$.

Defn: $\sum_{n=1}^{\infty} g_n(z)$ converges uniformly if $\forall \varepsilon > 0$
$$\exists M > 0 \text{ s.t. } N > M \underset{\text{implies}}{\Rightarrow} \left| \sum_{n=1}^{\infty} g_n(z) - \sum_{n=1}^{N} g_n(z) \right| < \varepsilon \quad \forall z \in D.$$

"I.e., the ε estimate for the error between the approximating sum and the limit is smaller than ε uniformly, i.e., for every $z \in D$ at once!"

Theorem: Assume $f(z) = \sum_{n=1}^{\infty} g_n(z)$ is finite for every $z \in D$, and the series converges absolutely and uniformly in D. Then
$$\int_{\mathcal{C}} f(z)\,dz \underset{\substack{\uparrow \\ \text{by definition}}}{=} \int_{\mathcal{C}} \sum_{n=1}^{\infty} g_n(z)\,dz \underset{\substack{\uparrow \\ \text{need to prove}}}{=} \sum_{n=1}^{\infty} \int_{\mathcal{C}} g_n(z)\,dz$$

Proof: Fix $\varepsilon > 0$. Uniform convergence implies

$$\exists\, M > 0 \text{ s.t. } N > M \underset{\text{implies}}{\Longrightarrow} \left| \sum_{n=1}^{N} g_n(z) - f(z) \right| < \varepsilon \quad \forall\, z \in D.$$

Thus:

can always pass summation thru finite sum

$$\Big| \underbrace{\sum_{n=1}^{N} \int_{\mathcal{C}} g_n(z)\,dz}_{I_N} - \underbrace{\int_{\mathcal{C}} f(z)\,dz}_{I} \Big|$$

$$= \left| \int_{\mathcal{C}} \left(\sum_{n=1}^{N} g_n(z) \right) dz - \int_{\mathcal{C}} f(z)\,dz \right|$$

$$= \left| \int_{\mathcal{C}} \left(\sum_{n=1}^{N} g_n(z) - f(z) \right) dz \right|$$

$$\leq |\mathcal{C}| \cdot \max_{z \in D} \Big| \underbrace{\sum_{n=1}^{N} g_n(z)}_{} - f(z) \Big|$$

The N'th approximation within ε of f for every z uniformly

$$\leq |\mathcal{C}| \cdot \varepsilon$$

Therefore $\lim_{N \to \infty} I_N = I$ by defn of convergence of a sequence of numbers in $\mathbb{C}$.

Thus:

$$\sum_{n=1}^{\infty} \int_{\mathcal{C}} g_n(z)\,dz = \lim_{N \to \infty} \sum_{n=1}^{N} \int_{\mathcal{C}} g_n(z)\,dz = \lim_{N \to \infty} I_N = I$$

$$= \int_{\mathcal{C}} f(z)\,dz = \int_{\mathcal{C}} \underbrace{\sum_{n=1}^{\infty} g_n(z)}_{f(z)}\,dz \quad \checkmark$$

Theorems we need to prove:

Theorem (Taylor) If f is analytic in some open set containing z_0, then

$$f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n \leftarrow \text{Power series}$$

with

$$c_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f(w)}{(w-z_0)^{n+1}} dw \leftarrow \text{Taylor Series}$$

Converges in some open ball $|z-z_0| < R$.
The largest R is called the radius of convergence.

Moreover, the Taylor Series converges absolutely and uniformly in $B_r(z_0)$ for all $r < R$, and diverges absolutely for all $r > R$.

Cor: R is the largest radius such that $f(z)$ extends to an analytic function in $B_R(z_0)$.
(I.e., the "distance to the nearest singularity in f")

Theorem (**Laurent**) If f is analytic in an annulus $r<|z-z_0|<R$, then

$$f(z) = \underbrace{\sum_{n=1}^{\infty} \frac{c_{-n}}{(z-z_0)^n}}_{\text{neg powers}} + \underbrace{\sum_{n=0}^{\infty} c_n (z-z_0)^n}_{\text{T-series terms}}$$

$$= \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$$

$c_{-n} = b_n$
$c_n = a_n$

Converges for all $r<|z-z_0|<R$, where

$(n \geq 1)$ $\quad c_{-n} = \frac{1}{2\pi i} \oint_C f(w)(w-z_0)^{n-1} dw \quad (a_n)$

$(n \geq 0)$ $\quad c_n = \frac{1}{2\pi i} \oint_C \frac{f(w)}{(w-z_0)^{n+1}} dw$ Same as Taylor ! $\quad (b_n)$

(Note: $c_{-1} = \frac{1}{2\pi i} \oint_C f(w)(w-z_0)^{1-1} dw = \frac{1}{2\pi i} \oint_C f(z)\,dz$

$\Rightarrow \oint_C f(z)\,dz = 2\pi i\, c_{-1}$)

Defn: The smallest r and the largest R st the Laurent Series converges in $r<|z-z_0|<R$ give the annulus of convergence

Cor: The Laurent series converges in the largest annulus in which f extends to an analytic function. I.e., "R and r extend to the nearest singularity in f".

Moreover, if $r < |z-z_0| < R$ is the annulus of convergence, then the Laurent Series converges absolutely and uniformly in every sub-annulus $r < r' < |z-z_0| < R' < R$, and diverges at every $z \notin \{z : r \leq |z-z_0| \leq R\}$, i.e., in complement of the closure of the annulus of convergence.

- In fact, the Weierstrass M-test will apply to both the pos and negative series in every sub-annulus, so we'll prove

$$|b_n (z-z_0)^n| \leq \overline{M}_n \quad \text{with} \quad \sum_{n=1}^{\infty} \overline{M}_n < \infty$$

and

$$|a_n (z-z_0)^n| \leq M_n \quad \text{with} \quad \sum_{n=0}^{\infty} M_n < \infty .$$

Proof of Taylors thm: Assume f is analytic in $B_R(z_0)$. First we show that if

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n, \quad \left(\text{i.e., "T-series converges"}\right)$$

and the convergence is uniform, then

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$

To see this, if all derivatives can be obtained by TxT differentiation, then

$$f^{(n)}(z) = \sum_{k=0}^{\infty} k(k-1)\cdots(k-n+1)\, a_k (z-z_0)^{k-n}.$$

Setting $z = z_0$, all terms vanish except $k=n$, so

$$f^{(n)}(z_0) = n!\, a_n \Rightarrow a_n = \frac{f^{(n)}(z_0)}{n!}.$$

Conclude: if the series & all TxT derivatives converge uniformly (so TxT differentiation justified) then the a_n must be the Taylor coefficients

$$a_n = \frac{f^{(n)}(z_0)}{n!} \overset{\text{CIF}}{=} \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f(w)}{(w-z_0)^{n+1}}\, dw \qquad \mathcal{C} \text{ winds once around } z_0$$

Thus we prove the Taylor series & all its TxT derivatives converge uniformly in $B_{R'}(z_0)$, $R' < R$.

- **To prove:** $\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$ converges absolutely and uniformly in $|z-z_0|<R'<R$, it suffices to prove $\sum_{n=0}^{\infty} \frac{|f^{(n)}(z_0)|}{n!}|z-z_0|^n$ converges uniformly, so by the Weierstrass M-test it suffices to find $M_n \in \mathbb{R}^+$ st $\sum_{n=0}^{\infty} M_n < \infty$ and $\frac{|f^{(n)}(z_0)|}{n!}|z-z_0|^n < M_n$ for every z st $|z-z_0|<R'<R$. By (CIF),

$$|f^{(n)}(z_0)| = \left|\frac{n!}{2\pi i}\oint_{C_{R'}} \frac{f(w)}{(w-z_0)^{n+1}}\,dw\right| \le \left|\frac{n!}{\cancel{2\pi} i}\right| \cancel{2\pi} R' \frac{M}{(R')^{n+1}} = \frac{M\,n!}{(R')^n}$$

$M = \max_{|z-z_0|\le R'} |f(z)|$

Thus:

$$\frac{|f^{(n)}(z_0)|}{n!}|z-z_0|^n \le \frac{M\,\cancel{n!}}{\cancel{(R')^n}}\frac{1}{\cancel{n!}}\cancel{(R')^n} = M = \text{constant}$$

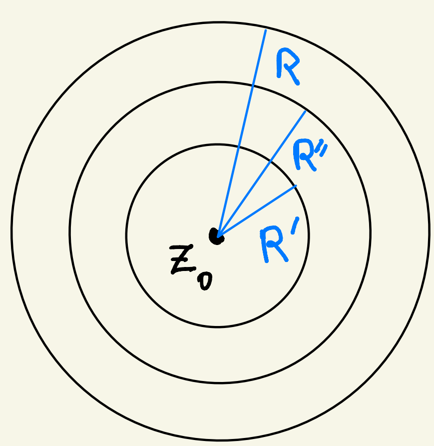
Note: $M_n = M$ is not small enough: $\sum_{n=1}^{N} M = NM \xrightarrow[N\to\infty]{} \infty$

- **Big Idea:** Choose $R''>R'$, $0<R'<R''<R$.

then: $\frac{R'}{R''}<1 \Rightarrow \sum_{n=0}^{\infty}\left(\frac{R'}{R''}\right)^n < 0$

Now - estimate $f^{(n)}(z_0)$ on $r=R''$

but estimate $|z-z_0|^n$ on $r=R'$!



That is:

$$|f^{(n)}(z_0)| = \left| \frac{n!}{2\pi i} \oint_{\underset{\sim}{\mathcal{C}_{R''}}} \frac{f(w)\,dw}{(w-z_0)^{n+1}} \right| \le \underbrace{\cancel{2\pi} R''}_{|\mathcal{C}_{R''}|} \underbrace{\frac{M}{\cancel{2\pi}} \frac{n!}{(R'')^{n\cancel{+1}}}}_{}$$

$M = \max_{|z-z_0| \le R''} |f(z)|$

$$\frac{|f^{(n)}(z_0)|}{n!} \underbrace{|z-z_0|^n}_{\substack{\text{estimate for} \\ |z-z_0| \le R'}} \le \frac{M \cancel{n!}}{(R'')^n} \cdot \frac{1}{\cancel{n!}} (R')^n = M \left(\frac{R'}{R''} \right)^n$$

"Use a smaller radius for $|z-z_0| \le R'$ than for $f^{(n)}(z_0) \Rightarrow |f^{(n)}(z_0)| \le \frac{Mn!}{(R'')}$

Now choose $M_n = \underset{\text{Const}}{M} \hat{r}^n$, $\hat{r} = \frac{R'}{R''} < 1$.

Then: $|a_n (z-z_0)^n| < M_n$, $\sum_{n=0}^{\infty} M_n = M \underbrace{\sum_{n=0}^{\infty} \hat{r}^n}_{\frac{1}{1-\hat{r}}} < \infty$ "geometric series"

Conclude: By Weierstrass M-test, the Taylor series converges absolutely and uniformly for $|z-z_0| \le R'$ for all $R' < R$ so long as f analytic in $B_R(z_0)$. Conclude – radius of convergence is largest R st f analytic in $B_R(z_0)$!

HW ① Let $\sum_{n=0}^{\infty} c_n$ be a series of complex numbers. Prove that if $\sum_{n=0}^{\infty} |c_n|$ is convergent as a series of <u>real</u> numbers, then $\sum_{n=0}^{\infty} c_n$ converges.

(Hint: Show $S_N = \sum_{n=0}^{N} c_n$ is Cauchy)

HW ② Prove that the <u>complex</u> geometric series $\sum_{n=m}^{\infty} z^n$ converges if $|z|<1$. In this case show $\sum_{n=m}^{\infty} z^n = \frac{z^m}{1-z}$ for all $|z|=r<1$.

(Hint: Same as Real case -

Let $S_N = \sum_{n=m}^{N} z^n \Rightarrow S_N - zS_N = z^m - z^{N+1} \longrightarrow z^m$.)

HW 3 Prove that the radius of convergence of $f'(z) = \sum_{k=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} n (z-z_0)^{n-1}$ is the same as the radius of convergence of $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)(z-z_0)^n}{n!}$.
Conclude (by induction) that the radius of convergence is the same for all derivatives of f. (Hint: Weierstrass M-test OR ratio test.)

HW 4 Use 3 to prove that you can differentiate a power series T×T inside its radius of convergence.

Idea: Let $g(z) = \sum_{n=0}^{\infty} n \frac{f^{(n)}(z_0)}{n!}(z-z_0)^{n-1}$ be the T×T derivative of $f(z)$. Since g converges uniformly on $B_R(z_0)$, $\int_C g(z)\,dz = \sum_{n=0}^{\infty} \int_C \frac{f^{(n)}(z_0)}{n!} n (z-z_0)^{n-1} dz$

$= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n = f(z)$ for any curve in $B_R(z_0)$ taking z_0 to z. Conclude: $f(z)$ is an anti-derivative of g $\Rightarrow$ $f'(z) = g(z)$ ✓

Proof of Theorem (Laurent): Assume f is analytic in annulus $r<|z-z_0|<R$. We show Laurent series $f(z)=\sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$ converges absolutely and uniformly in $r'<|z-z_0|<R'$ for all $r<r'<R'<R$. We first show that, assuming we can integrate T×T thru the Σ-sign of Laurent Series, we get the formulas (a_n), (b_n) of the Theorem (Laurent).

I.e., writing $b_n = c_{-n}$, $a_0 = c_0$, $a_n = c_n$, $n=1,2,3\ldots$

$$f(z)=\sum_{n=-\infty}^{\infty} c_n (z-z_0)^n = \sum_{n=1}^{\infty} b_n (z-z_0)^{-n} + \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

$\Rightarrow$ (for any scc C winding clockwise once around annulus)

$$\oint_C f(w)\,dw = \sum_{n=-\infty}^{\infty} c_n \oint_C \underbrace{(w-z_0)^n}_{} dw = c_{-1}\, 2\pi i$$

all powers have anti-deriv's except $n=-1$!

$$\Rightarrow \boxed{b_1 = c_{-1} = \frac{1}{2\pi i}\oint_C f(w)\,dw}$$

Similarly, we get a formula for all a_n by multiplying f by negative powers of $(z-z_0)$ and integrating:

$$f(z)(z-z_0)^{-1} = \underbrace{\sum_{n=1}^{\infty} b_n (z-z_0)^{-n-1}}_{\text{starts at power } -2 \Rightarrow \text{ all powers have anti-deriv}} + \frac{a_0}{z-z_0} + \underbrace{a_1 + a_2(z-z_0) + \cdots}_{\text{all positive powers have anti-derivative}}$$

$$\oint_{\mathcal{C}} f(w)(w-z_0)^{-1}\,dz = a_0 2\pi i \Rightarrow \boxed{a_0 = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{f(w)}{w-z_0}\,dz}$$

Continuing, we get formula for all a_n, $n \geq 0$:

$$\oint_{\mathcal{C}} f(z)(z-z_0)^{n}\,dz = a_{n-1} 2\pi i \Rightarrow \boxed{a_n = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{f(w)}{(w-z_0)^{n+1}}\,dw}$$

These are the same as the formulas in T-series, except they do not give a value to $f^{(n)}(z_0)$ because $f(z)$ is not analytic at $z = z_0$!

Again: $\mathcal{C}$ is any SCC in annulus $\overline{B_{R'}(z_0)} \setminus B_{r'}(z_0)$ which winds once around $z = z_0$, counter-clockwise.

Similarly, we get integral formula for all the b_n, assuming we can integrate T×T:

$$f(z)=\sum_{n=-\infty}^{\infty} c_n (z-z_0)^n = \sum_{n=1}^{\infty} b_n (z-z_0)^{-n} + \sum_{n=0} a_n (z-z_0)^n$$

$$f(z)(z-z_0) = \underbrace{b_1}_{} + \frac{b_2}{z-z_0} + \underbrace{\sum_{n=3}^{\infty} \frac{b_n}{(z-z_0)^{n-3}}}_{} + \underbrace{\sum_{n=0} a_n (z-z_0)^n}_{}$$

all have anti-deriv's ⇒ 0-integrals

$$\oint_{\mathcal{C}} f(w)(w-z_0)\,dw = b_2\, 2\pi i \Rightarrow \boxed{b_2 = \frac{1}{2\pi i}\oint_{\mathcal{C}} f(w)(w-z_0)\,dz}$$

$$\vdots$$

$$\oint_{\mathcal{C}} f(w)(w-z_0)^{n-1}\,dw = b_n\, 2\pi i \Rightarrow \boxed{b_n = \frac{1}{2\pi i}\oint_{\mathcal{C}} f(w)(w-z_0)^{n-1}\,dw}$$

Summary: Assuming uniform convergence

$$a_n = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{f(w)}{(w-z_0)^{n+1}}\,dw, \quad b_n = \frac{1}{2\pi i}\oint_{\mathcal{C}} f(w)(w-z_0)^{n-1}\,dw$$

or

$$\boxed{c_n = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{f(w)}{(w-z_0)^{n+1}}\,dw} \qquad n\in\mathbb{Z}=\{0,\pm1,\pm2,\pm3,\dots\}$$

integers

Thus it remains only to prove the Laurent Series converges uniformly in annulus $r' \le |z-z_0| \le R'$, $\left(\overline{B_{R'}(z_0) \setminus B_{r'}(z_0)}\right)$.

For this we estimate a_n on $C_{R''}$ $(n \ge 0)$ and b_n on $C_{r''}$ $(n \ge 1)$, $r < r'' < r' < R' < R'' < R$

$$|a_n| = \left| \frac{1}{2\pi i} \oint_{C_{R''}} \frac{f(w)}{(w-z_0)^{n+1}} dw \right| \le \frac{M}{2\pi} |C_{R''}| \frac{1}{(R'')^{n+1}}$$

$$\le M \frac{1}{(R'')^n} \quad \left(\text{as before}\right)$$

$$|b_n| = \left| \frac{1}{2\pi i} \oint_{C_{r''}} f(w) (w-z_0)^{n-1} dw \right|$$

$$\le \frac{1}{2\pi} |C_{r''}| M (r'')^{n-1} \le M (r'')^n$$

Thus in $r' \le |z-z_0| \le R'$ we have

$$|a_n (z-z_0)^n| \le M \left(\frac{R'}{R''}\right)^n$$

$$\Rightarrow |c_n (z-z_0)^n| \le M \hat{r}^n$$

$$|b_n (z-z_0)^{-n}| \le M \left(\frac{r''}{r'}\right)^n$$

$$\hat{r} = \operatorname{Max}\left\{\frac{R'}{R''}, \frac{r''}{r'}\right\} < 1.$$

Conclude: Choosing $M_n = M\hat{r}^n$, we have

$$|a_n||z-z_0|^n \le M_n \qquad \sum_{n=0}^{\infty} M_n < \infty$$
$$|b_n||z-z_0|^{-n} \le M_n$$

$\Rightarrow$ $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges uniformly $r' \le |z-z_0| \le R'$

$\sum_{n=1}^{\infty} b_n(z-z_0)^{-n}$ converges uniformly $r' \le |z-z_0| \le R'$

$\Rightarrow$ Laurent Series

$$\sum_{n=-\infty}^{\infty} c_n(z-z_0)^n = \sum_{n=1}^{\infty} b_n(z-z_0)^{-n} + \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

converges <u>uniformly</u> in annulus $r' \le |z-z_0| \le R'$.

Since this works so long as $r' > r$, $R' < R$ and f analytic in $r < |z-z_0| < R$, it follows that the max'al annulus of convergence is the largest annulus in which f is analytic!

HW 4 Prove the T×T derivative of the Laurent Series has the same annulus of convergence $r<|z-z_0|<R$ as the Laurent series itself. 20

5 Prove the T-series and Laurent Series Diverge outside the radius (or annulus) of convergence. (Hint: if the series converges absolutely in a larger radius than the radius of convergence R (or annulus of convergence $r<|z-z_0|<R$), then R wasn't the true radius of convergence.)

Classification of Singularities:

Defn: z_0 is an isolated singularity of f if f is analytic in some deleted nbhd of z_0, i.e., f analytic in $B_\varepsilon(z_0) \setminus \{z_0\}$

Defn: If z_0 is an isolated singularity of f and its Laurent expansion has only a finite number of non-zero b_n's such $b_n = c_{-n} \neq 0$, (or alternatively, $c_n = 0$ for all but a finite # of $n<0$), then we say z_0 is a pole of f. I.e., if $c_{-N} \neq 0$ and

$$f(z) = \underbrace{\sum_{n=1}^{N} b_n (z-z_0)^{-n}} + \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

$b_N = c_{-N} \neq 0$ so N is largest neg st $c_{-N} \neq 0$

then we say z_0 is a pole of order N.

Defn: If $c_{-n} \neq 0$ for arbitrarily large n, then we say f has an essential singularity at $z = z_0$.

Defn: A function which is analytic except at a finite number of poles is called a meromorphic function.

We will prove:

Theorem: Rational functions $f(z) = \frac{p(z)}{q(z)}$, which are the ratio of polynomials p and q, are always meromorphic. So are $f(z) = g(z) \cdot \frac{p(z)}{q(z)}$ where g is analytic.

Essential singularities are very complicated as express in following theorem which is the topic of a more advanced class →

Theorem (**Picard's Thm**) An analytic function takes on every complex value (except possibly one) in every nbhd of an isolated essential singularity.

Examples of analytic functions with essential singularities at isolated points –

$$f(z) = e^{1/z}, \ \sin\frac{1}{z}, \ z^2 \sin\frac{1}{z}, \ \text{etc.}$$

We restrict to the study of poles.

HW 6 Show $f(z) = \dfrac{1}{e^z - 1}$ has a pole of order 1 at $z=0$.

(Hint: $f(z) = \dfrac{1}{z\left(1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots\right)} = \dfrac{1}{z} g(z)$ where $g(z)$ is analytic at $z=0$. $\Rightarrow g(z)$ has T-series ...)

Examples: (Most important) We now give the simplest examples which display the ideas behind T-series, L-Series and the Residue Theorem.

- Consider First: $f(z) = \frac{1}{1-z}$, analytic for $z \neq 1$.

To construct T-Series about $z_0 = 0$ we could calculate $\frac{f^{(n)}(0)}{n!}$; but easier to view $\frac{1}{1-z}$ as the limit of a geometric Series —

$$f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

Since T-series is unique this must be it!

For $z = x = x + 0\cdot i$ real, this reduces to

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

Recall how to sum a geometric series $\sum_{n=0}^{\infty} z^n$:

$$S_N = \sum_{n=0}^{N} z^n = 1 + z + \cdots + z^N.$$

$$zS_N = \sum_{n=0}^{N} z^{n+1} = z + z^2 + \cdots + z^N + z^{N+1}$$

Subtract: $(1-z)S_N = 1 - z^{N+1} \Rightarrow S_N = \frac{1-z^{N+1}}{1-z} \longrightarrow \frac{1}{1-z}$ for $|z|<1$.

Conclude: $\sum_{n=0}^{\infty} z^n$ is the T-series for $f(z) = \frac{1}{1-z}$ expanded about $z_0 = 0$, and it converges for $|z - 0| < R$ where R = distance from z_0 to the nearest singularity in f, namely, $z = 1$. Thus the distance from $z=0$ to $z=1$ is $\boxed{R=1}$!

Thus $f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ for $|z| < 1$. Note $f(z)$ extends as analytic around the singularity, i.e. $f(z) = \frac{1}{1-z}$ is analytic $\forall\, z \in \mathbb{C}$ except $z = 1$ by the Quotient Rule – but values of f beyond $|z| = 1$ will not be given by the Taylor Series expansion about $z = 0$!

- Consider Next: $f(z) = \frac{1}{1+z^2}$ analytic for $z \neq \pm i$.

Using the geometric series we can get the T-series expansion about $z = 0$:

$$f(z) = \frac{1}{1+z^2} \underset{\uparrow}{=} \sum_{n=0}^{\infty} (-z^2)^n = \sum_{n=0}^{\infty} (-1)^n z^{2n}.$$

View as the sum of a geometric series

Converges for $|z|^2 < 1 \Leftrightarrow |z| < 1 \Rightarrow R = 1$.

Thus: $z \in \mathbb{C}$ $\quad f(z) = \frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-1)^n z^{2n}$

$z = x \in \mathbb{R}$ $\quad f(x) = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$

Note: Complex Variables explains why $R=1$

Namely, the distance from $z_0 = 0$ to the nearest singularties $z = \pm i$ is $R = 1$.

However: If all we had is the real function $f(x) = \frac{1}{1+x^2}$, we don't know why $R=1$!

Conclude: Complex Power series explain why Real power series converge!

I.e., ahead of time, how would one guess that $i = \sqrt{-1}$ would explain why $f(x) = \frac{1}{1+x^2}$ is summable? (Complex Variables is magic!)

- We now calculate the residues of $f(z) = \frac{1}{1+z^2}$ at $z = \pm i$. I.e., we find $R(f,i)$ & $R(f,-i)$, the coeff $c_{-1} = b_1$ of the Laurent expansion of $f(z) = \frac{1}{1+z^2}$ about singularities $z = i$ and $z = -i$.

We first evaluate $R(f, i)$, the residue @ $z = i$:

Without computing it, our theory tells us $\exists$ a Laurent Expansion of $f(z) = \frac{1}{1+z^2}$ about $z = i$, and it converges for $|z-i| < R = 2$ since the nearest singularity is located at $z = -i$, and distance from $z = i$ to $z = -i$ is $|i-(-i)| = |2i| = 2$.

Thus without computing them, we know c_n exist such that for $|z-i|<2$ we have

$$f(z) = \frac{1}{1+z^2} = \sum_{n=1}^{\infty} \frac{c_{-n}}{(z-i)^n} + \sum_{n=0}^{\infty} c_n (z-i)^n$$

To evaluate $c_{-1} = b_1$, note $f(z)(z-i) = \frac{1}{z+i}$ is analytic in $B_2(i)$, but also

$$\frac{1}{z+i} = f(z)(z-i) = \underbrace{\sum_{n=1}^{\infty} \frac{c_{-n}}{(z-i)^{n-1}}}_{\cdots + \frac{c_{-3}}{(z-i)^2} + \frac{c_{-2}}{(z-i)} + c_{-1}} + \sum_{n=0}^{\infty} c_n (z-i)^{n+1} \quad (*)$$

Setting $z = i$ on LHS gives $\frac{1}{2i} = -\frac{i}{2}$.

Now observe: to get a finite value on RHS there cannot be any negative powers $(z-z_0)^{-n}$

$\Rightarrow c_{-n} = b_n = 0$ $n \geq 2$; and positive powers $(z-i)^n$ vanish @ $z = i$!

Conclude: The fact that

$$\lim_{z\to i} f(z)(z-i) = -\frac{i}{2} \quad \text{finite}$$

implies $c_{-n}=0$ all $n\geq 2$, and so the RHS of (*) gives

$$-\frac{i}{2} = \lim_{z\to i}\left\{ c_{-1} + \sum_{n=0}^{\infty} c_n \underbrace{(z-i)^{n+1}}_{0} \right\} = c_{-1}$$

We have:

$$\boxed{c_{-1} = \lim_{z\to i} f(z)(z-i) = -\frac{i}{2}}$$

Note: We never used the Laurent Series at all, except knowing it exists!

Conclude: $f(z) = \frac{1}{1+z^2}$ has a pole of order 1 at $z=i$, and $R\left(\frac{1}{1+z^2}, i\right) = -\frac{i}{2}$.

This is the main idea for calculating residues at poles, when only a finite number of negative powers c_{-n} in L-series are nonzero!

- More generally: assume $f(z)$ has a pole of order n at $z=z_0$. Then

$$f(z) = \sum_{k=1}^{n} \frac{c_{-k}}{(z-z_0)^k} + \sum_{k=0}^{\infty} c_k (z-z_0)^k$$

and so

$$f(z)(z-z_0)^n = c_{-n} + c_{-(n-1)}(z-z_0) + \cdots + c_{-1}(z-z_0)^{n-1} + \sum_{k=0}^{\infty} c_k (z-z_0)^k$$

and evaluating at $z=z_0$ gives

(*) $\lim_{z\to z_0} f(z)(z-z_0)^n = c_{-n}$ which implies

$\lim_{z\to z_0} f(z)(z-z_0)^k = c_{-k} = 0$ for $k>n$.

Moreover, (*) is all we need to conclude $f(z)$ has a pole of order n at $z=z_0$.

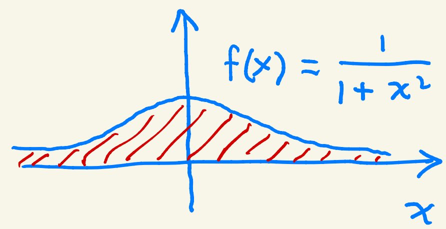
We record this as a theorem:

Theorem: If $f(z)$ is analytic in a deleted nbhd of $z=z_0$, then f has a pole of order n iff (*) holds. Moreover, if $\lim_{z\to z_0} f(z)(z-z_0)^n = 0$, then z_0 is a pole of order $< n$.

We'll think about how to calculate residues for poles of order $n \geq 2$ after doing this

fundamental example:

- **Example:** Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$



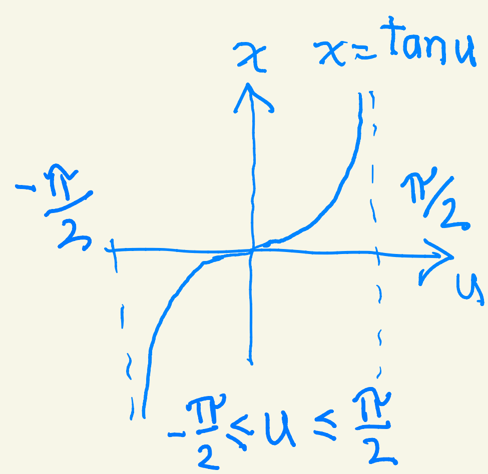
(Note: $\exists$ finite area under graph because $f(x) \sim \frac{1}{x^2}$ as $x \to \pm\infty$, i.e. $\underbrace{\displaystyle\int_1^\infty \frac{dx}{x^\alpha} = \frac{x^{-\alpha+1}}{-\alpha+1}\Big]_1^\infty}_{\text{finite if } \alpha > 1} = 0 - \left(\frac{1}{-\alpha+1}\right) = \frac{1}{\alpha-1}$)

Substitution works in this case & we use it **as a check:**

$x = \tan u$, $dx = \sec^2 u\, du \Rightarrow$

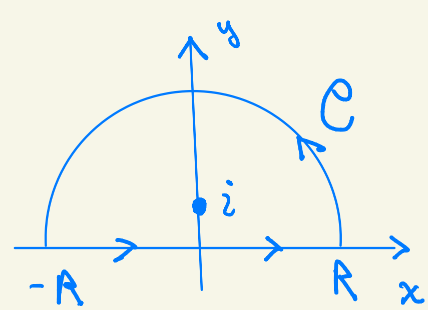
$-\frac{\pi}{2} \leq u \leq \pi/2$

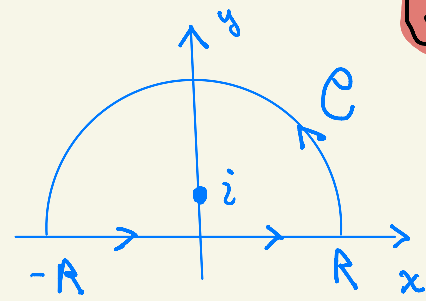
$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\pi/2}^{+\pi/2} \frac{\sec^2 u}{\sec^2 u}\,du = \pi$$



- **Alternatively**, we can use **complex variables**:

Consider the sce $\mathcal{C}$ consisting of the upper semi-circle of radius R, and the closed interval $[-R, R]$, and write $\mathcal{C} = \mathcal{C}_R + \mathcal{C}_{[-R,R]}$





- Then $\mathcal{C}$ is a simple closed positively oriented curve which goes around the singularity $z_0 = i$ of $f(z) = \frac{1}{1+z^2}$ exacty once. Thus the Residue Thm tells us:

$$\oint_{\mathcal{C}} f(z)\,dz = 2\pi i\, R(f, i)$$

$$R(f,i) = \lim_{z \to z_0} f(z)(z - z_0) = \lim_{z \to i} \frac{1}{(z-\cancel{i})(z+i)} (\cancel{z-i}) = \frac{1}{2i}$$

Conclude: $\oint_{\mathcal{C}} \frac{dz}{1+z^2} = \cancel{2}\pi \cancel{i}\left(\frac{1}{\cancel{2i}}\right) = \pi$

- Now note: on $\mathcal{C}_R$, $|f(z)| = \left|\frac{1}{1+z^2}\right| \leq \frac{1}{R^2-1}$.

So $\int_{\mathcal{C}_R} \frac{dz}{1+z^2} \leq |\mathcal{C}_R| \frac{1}{R^2-1} = \frac{\pi R}{R^2-1} \xrightarrow{R\to\infty} 0$

and $\int_{\mathcal{C}_{[-R,R]}} \frac{dz}{1+z^2} \underset{\uparrow}{=} \int_{-R}^{R} \frac{dx}{1+x^2}$

z real on $\mathbb{R}$

Thus $\forall R > 0$ we have:

$$\pi = \int_{\mathcal{C}} \frac{dz}{1+z^2} = \underbrace{\int_{\mathcal{C}_R} \frac{dz}{1+z^2}}_{\text{tends to zero}} + \underbrace{\int_{-R}^{R} \frac{dx}{1+x^2}}_{\text{tends to } \int_{-\infty}^{\infty} \frac{dx}{1+x^2}}$$

I.e. as $R \to \infty$

Taking the limit $R \to \infty$ gives

$$\pi = \lim_{R \to \infty} \int_{-R}^{R} \frac{dx}{1+x^2} = \int_{-\infty}^{\infty} \frac{dx}{1+x^2} \quad \checkmark$$

Defn of Improper Integral

- HW 7 Do argument using the lower semicircle $\mathcal{C}_{-R} - \mathcal{C}_{[-R,R]}$.

- In many important examples the Residue Thm works but Real Substitution does not work.

- It's also of theoretical importance because values of real integrals can be represented and analyzed mathematically as residues of complex valued functions...

☒ Note: that our theory tells us that every analytic function $f(z)$ can be expanded in Laurent Series in any annulus in which it is analytic - and we have an integral formula for each c_n, the coefficient of the power $(z-z_0)^n$ in the L-expansion, $n \in \mathbb{Z}$. integers

However, the integral formulas alone do not evaluate the residue c_{-1} directly:

Specifically - recall L-expansion:

$$f(z) = \sum_{k=2}^{\infty} \frac{c_{-k}}{(z-z_0)^k} + \underbrace{\frac{c_{-1}}{z-z_0}}_{c_{-1} = R(f, z_0)} + \sum_{k=0}^{\infty} c_k (z-z_0)^k$$

with integral formulas (valid for all $n \in \mathbb{Z}$)

$$c_n = \frac{1}{2\pi i} \oint_C \frac{f(w)}{(w-z_0)^{n+1}} dw.$$

But formula for c_{-1} just reproduces Residue Thm:

$$c_{-1} = \frac{1}{2\pi i} \oint_C f(w)\,dw \Rightarrow \underbrace{\oint_C f(z)\,dz = 2\pi i\, c_{-1}}_{\text{Residue Thm}}.$$

Provides no information about value of c_{-1}!

Instead of integral formula:

Simple Pole: $C_{-1} = \lim_{z \to z_0} f(z)(z-z_0)$.

What about double pole? Pole of order n?

- **Double Pole:**

$$f(z) = \frac{C_{-2}}{(z-z_0)^2} + \frac{C_{-1}}{(z-z_0)} + \sum_{k=0}^{\infty} C_k (z-z_0)^k$$

- $\lim_{z \to z_0} f(z)(z-z_0)^2 = C_{-2} \neq 0$

$\Rightarrow \lim_{z \to z_0} f(z)(z-z_0)^k = 0 \quad k = 3, 4, 5, \ldots$

- Obtain C_{-2} by $C_{-2} = \lim_{z \to z_0} f(z)(z-z_0)^2$
- To get C_{-1}, write

$$g(z) = f(z) - \frac{C_{-2}}{(z-z_0)^2} \Rightarrow g \text{ has simple pole w same } C_{-1}$$

- Conclude: $C_{-1} = \lim_{z \to z_0} g(z)(z-z_0)$.

- **Example**: $f(z) = \dfrac{1}{(z-i)^2(z+i)}$ Find $R(f, i)$

$z_0 = i$

① $\lim\limits_{z\to i} f(z)(z-i)^2 = \dfrac{1}{2i} = C_{-2}$

$\Rightarrow \lim\limits_{z\to i} f(z)(z-i)^k = 0 \ \forall k > 2$

$\Rightarrow$ double pole

② $g(z) = f(z) - \dfrac{1}{2i}\dfrac{1}{(z-i)^2}$ has simple pole with same C_{-1}

$\Rightarrow C_{-1} = \lim\limits_{z\to i} g(z)(z-i)$

$$g(z)(z-i) = \left(f(z) - \frac{1}{2i}\frac{1}{(z-i)^2}\right)(z-i) = f(z)(z-i) - \frac{1}{2i(z-i)}$$

$$= \frac{1}{(z-i)^2(z+i)}(z-i) - \frac{1}{2i(z-i)}$$

$$= \frac{1}{(z-i)(z+i)} - \frac{1}{2i(z-i)} = \frac{2i-(z+i)}{2i(z-i)(z+i)}$$

$$= \frac{-\cancel{(z-i)}}{2i\cancel{(z-i)}(z+i)} = \frac{-1}{2i(z+i)}$$

So: $C_{-1} = \lim\limits_{z\to i} g(z)(z-i) = \lim\limits_{z\to i} \dfrac{-1}{2i(z+i)} = \dfrac{-1}{(2i)^2} = \dfrac{1}{4}$

Conclude: $\boxed{R\left(\dfrac{1}{(z-i)^2(z+i)}, i\right) = \dfrac{1}{4}}$

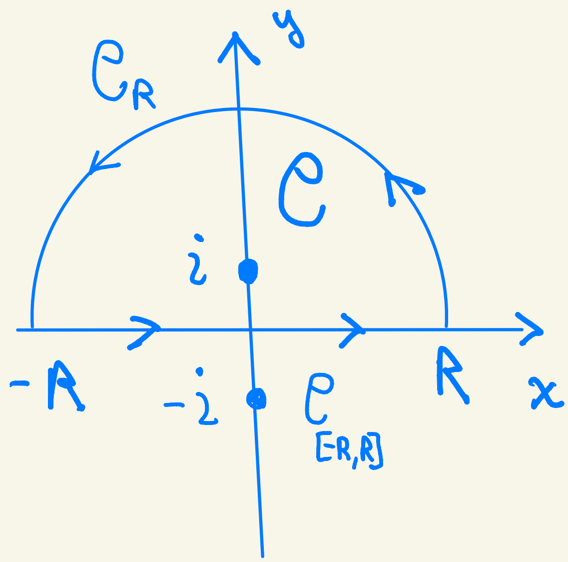
- **Application:** Let $\mathcal{C}$ be any scc which encloses $z_0 = i$ but not $z = -i$. Then

$$\oint_{\mathcal{C}} \frac{dz}{(z-i)^2(z+i)} = 2\pi i \, \underbrace{C_{-1}}_{C_{-1} = \frac{1}{4}} = \frac{\pi}{2} i$$

(HW) 8 **Evaluate** $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2}$

Hint: $f(z) = \dfrac{1}{(z-i)^2(z+i)^2}$

$(z-i)(z+i) = z^2+1$



- Consider $\mathcal{C} = \mathcal{C}_R + \mathcal{C}_{[-R,R]}$

$$\oint_{\mathcal{C}} \frac{dz}{(z-i)^2(z+i)^2} = \int_{\mathcal{C}_R} + \int_{\mathcal{C}_{[-R,R]}} = 2\pi i \, \underbrace{R(f, i)}_{\text{You need to evaluate this!}}$$

$$\left| \int_{\mathcal{C}_R} \frac{dz}{(z-i)^2(z+i)^2} \right| \leq 2\pi R \frac{1}{(R^2-1)(R^2-1)} \xrightarrow[R\to\infty]{} 0$$

$$\big||z| - |w|\big| \leq |z+w| \leq |z| + |w|$$

$$\lim_{R\to\infty} \int_{\mathcal{C}_{[-R,R]}} \frac{dz}{(z-i)^2(z+i)^2} \underset{\substack{z = x \\ dz = dx}}{=} \lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{(x^2+1)^2} = \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2}$$

etc.