CH I – CAUCHY'S Thm & FTC

Math 185A
Winter 2022

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[1] Math 185A : Complex Variables

Instructor : Blake Temple

Research: General Relativity & Shock Wave Theory

2 midterms & final exam.

HW assigned but not collected. Soln's online.

Syllabus – Follow Dept Recommendations { But I may not follow or cover everything in text

[2] Introduction to Complex Variables –

- One can learn the Residue Thm & How to ~~learn~~ to use the machinery – I always wanted to know how one might discover the theory / How do you intuit the principles that underlye the theory? Partly this comes from understanding the proofs.
- So... what is complex variables, & how did the subject get started –
- Short Answer – How to incorporate $i = \sqrt{-1}$ into the theory of calculus?

- History: the subject was written by

Augustin Cauchy 1789-1857

Peter Dirchlet 1805-1859

Karl Weierstrass 1815-1897

Georg Friedrich Bernhard Riemann 1826-1866 *

- Motivation / Background

By letting $i = \sqrt{-1}$, you can factor polynomials

Eg: $x^2+1 \neq 0$ no real factors - irreducible polyn

$x = \pm i \quad (\pm i)^2 + 1 = -1+1 = 0$ ✓

$\therefore (x^2+1) = (x+i)(x-i)$ factors.

[Eg $P(x_0) = 0 \Rightarrow (x-x_0)$ is a factor (FIP)]

You can prove that all polyn's with real coeff's can be factored into linear & irreducible quad factors like $x^2+1$, $x^2+x+1$, etc.

Conclude - if all quadratic polyn's can be factored, then all polyn's can be factored into linear factors using $i = \sqrt{-1}$ ; ie

$$ax^2 + bx + c = 0 \qquad x_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

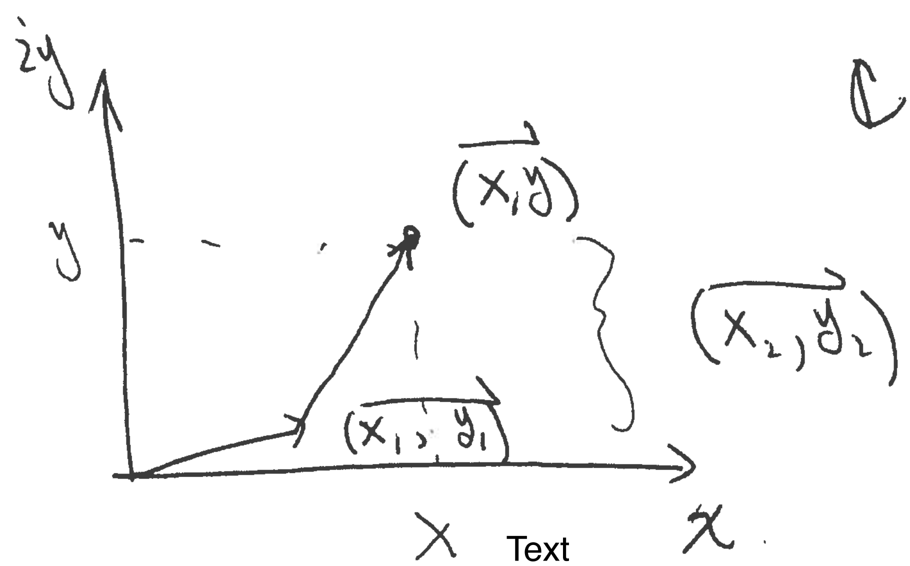
if $b^2 > 4ac$ then real factors

if $b^2 < 4ac$ then $x_{\pm} = \dfrac{-b \pm \sqrt{(4ac - b^2)}\, i}{2a}$

- So what is $i$ ?

General complex number $\underset{\text{real part}}{x} + i\underset{\text{complex part}}{y} = z$

So really - complex numbers are just ordered pairs $(x, y) \in \mathbb{R}^2$




$\mathbb{C} \equiv$ complex plane $\equiv \mathbb{R}^2$

$$(x_1 + i y_1) + (x_2 + i y_2) = (x_1 + x_2) + (y_1 + y_2) i$$

<u>Conclude</u>. adding (and scalar multiplying) components is the same as adding (and scalar multiplying) vectors in $\mathbb{R}^2$.

<u>Conclude</u>. $\mathbb{C}$ is just the vector space $\mathbb{R}^2$ together with a way to <u>multiply</u> vectors:

$\mathbb{R}^2$ $\quad (x_1, y_1) \cdot (x_2, y_2) = ?$ not defined

$\mathbb{C}$ $\quad (x_1 + i y_1) \cdot (x_2 + i y_2) = (x_1 x_2 - y_1 y_2) + (x_2 y_1 + x_1 y_2) i$

"$i$ gives compact way to define multiplication"

- In fact: this multiplication meets all the requirements of a <u>field</u> FIP

Field Axioms –

Field $\Rightarrow$ "all you need to do standard algebra"

| | Add | Mult |
|---|---|---|
| comm | $z+w=w+z$ | $z\cdot w=w\cdot z$ |
| assoc | $(z+w)+s=z+(w+s)$ | $(z\cdot w)s=z\cdot(w\cdot s)$ |
| add. id / mult id | $z+0=z$ | $z\cdot 1=z \quad z\neq 0$ |
| add inv. / mult inv | $(-z)+z=0$ | $\left(\frac{1}{z}\right)z=1 \quad z\neq 0$ |

Dist Prop

$$z(w+s)=z\cdot w+z\cdot s$$

All field axioms are met with

$$z_1+z_2=(x_1+x_2)+(y_1+y_2)i$$

$$z_1\cdot z_2=(x_1+iy_1)(x_2+iy_2) \qquad i^2=-1$$

We say $\mathbb{C}$ is a field that extends $\mathbb{R}$

$$z=x+iy\in\mathbb{C}$$

extends: $x\equiv x+0\cdot i\in\mathbb{R}$

<u>Ex</u>: $z=x+iy \quad z^{-1}=\frac{1}{z}=\frac{1}{x+iy}\frac{(x-iy)}{(x-iy)}=\frac{x}{x^2+y^2}+i\frac{-y}{x^2+y^2}$

"all formulas follow from field axioms together with "$i^2=-1$"

- What $\mathbb{R}$ has that $\mathbb{C}$ doesn't is an ordering:

$$x, y \in \mathbb{R} \Rightarrow x < y,\ x > y \text{ or } x = y$$

No such ordering of $\mathbb{C}$ or $\mathbb{R}^2$

- So... $\mathbb{C}$ is just $\mathbb{R}^2$ with a multiplication which makes it into a field.

- But... we can also talk about <u>convergence</u> in $\mathbb{R}^2$ & $\mathbb{C}$ (same notion)

In $\mathbb{R}^2$: $\underset{\sim}{x} = (x, y)$ $\quad \underset{\sim}{x}_n = (x_n, y_n)$

$$\underset{\sim}{x}_n \xrightarrow{n\to\infty} \underset{\sim}{x}_0 \quad \text{iff} \quad \|\underset{\sim}{x}_n - \underset{\sim}{x}_0\| \xrightarrow{n\to\infty} 0$$

$$\Leftrightarrow \sqrt{(x_n - x_0)^2 + (y_n - y_0)^2} \xrightarrow{n\to\infty} 0$$

$$\|(x, y)\|^2 = x^2 + y^2$$

In $\mathbb{C}$: $z = x + iy$, $|z|^2 = x^2 + y^2$ $\quad |z| = \sqrt{x^2 + y^2}$

We say $z_n \xrightarrow{n\to\infty} z$ iff $|z_n - z| \xrightarrow{n\to\infty} 0$ $\quad$ Same in $\mathbb{C}$ as in $\mathbb{R}^2$

- Conclude. Convergence in $\mathbb{C}$ is the same as convergence in $\mathbb{R}^2$

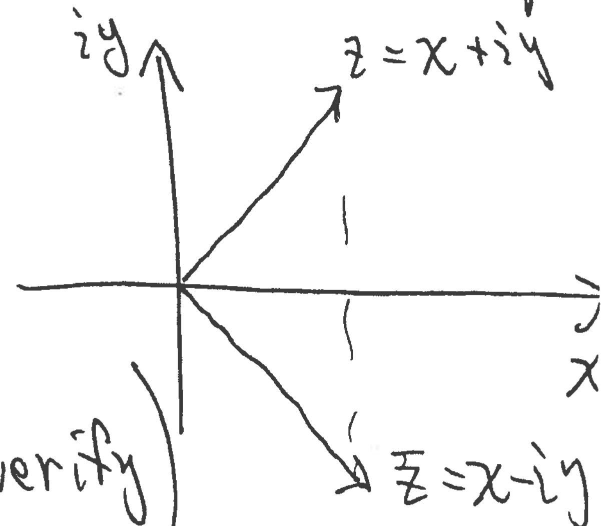
The only difference betw $\mathbb{R}^2$ & $\mathbb{C}$ is that you can multiply vectors in $\mathbb{C}$

Note: For $n > 2$, there is no multiplication that makes $\mathbb{R}^n$ into a field ! So $\mathbb{C}$ is special

(quaternions are not commutative in $\mathbb{R}^3$)

- **Note**: $|z|^2 = z \cdot \bar{z} = \underbrace{(x+iy)}_{z}\underbrace{(x-iy)}_{\bar{z}} = x^2 + y^2$

**Defn**: $z = x+iy \Rightarrow \bar{z} = x - iy$ is the complex conjugate of $z$




- **Thm**: (FIP)

① $\overline{zw} = \bar{z}\,\bar{w}$ (use field axioms to verify)

$\overline{z+w} = \bar{z} + \bar{w}$

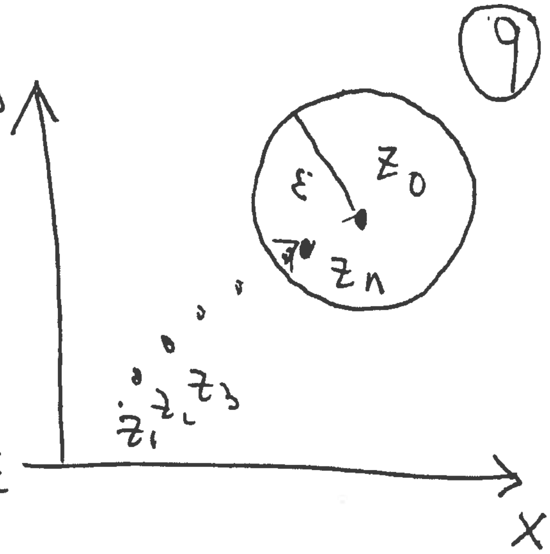
② $\frac{1}{2}(z + \bar{z}) = x \in \mathbb{R}$

$\frac{1}{2}(z - \bar{z}) = iy$ pure imaginary

③ $|z| > 0$ unless $z = 0$

- **Example**: Assume

$$z_n \xrightarrow{n\to\infty} z_0$$




$|z - z_0| < \varepsilon \iff \|(x,y) - (x_0,y_0)\| < \varepsilon$

$\iff \sqrt{(x-x_0)^2 + (y-y_0)^2} < \varepsilon$

$\iff$ distance from $(x,y)$ to $(x_0,y_0) < \varepsilon$

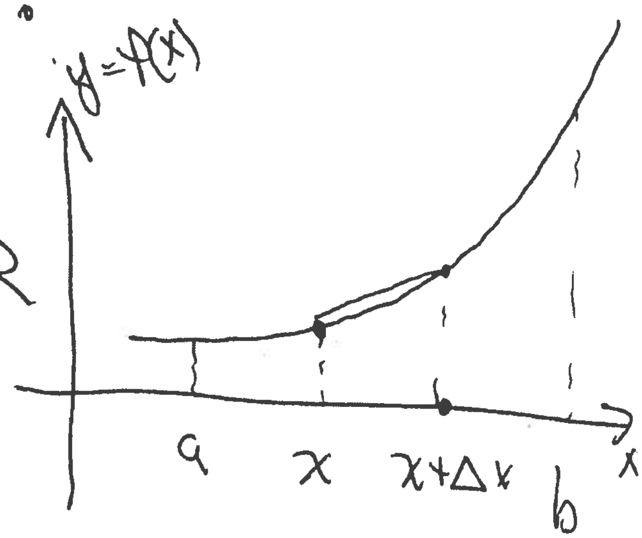
Conclude: $\{z : |z - z_0| < \varepsilon\}$ = set of all $z$ lying within the disc of radius $\varepsilon$, center $z_0$

$B_\varepsilon(z_0) = \{z : |z - z_0| < \varepsilon\}$ = "open ball of radius $\varepsilon$"

Defn: $z_n \to z_0$ iff $\forall \varepsilon\ \exists N$ such that if $n > N$, then $|z_n - z_0| < \varepsilon$

"For every $\varepsilon$ there exists an $N$ such that beyond $N$, $z_n$ lies in the ball of radius $\varepsilon$ center $z_0$"

⑤ Since in $\mathbb{C}$ we can add, subtr, mult, divide & take limits — we can do calculus just like we can in $\mathbb{R}$:




<u>Review</u>: $\mathbb{R}$ $\quad y = f(x)$

$f : \mathbb{R} \to \mathbb{R}$

$$f'(x) = \underbrace{\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}}$$

all you need is subtr, divide & take limit

$$I = \int_a^b f(t)\,dt \quad = \text{area under } f \text{ betw } a, b$$

$$F(x) = \int_a^x f(t)\,dt \quad \text{is an anti-derivative of } f$$

$$F'(x) = f(x)$$

FTC: $\displaystyle\int_a^b f(t)\,dt = F(b) - F(a)$ where $F'(x) = f(x)$

"areas can be evaluated by anti-derivatives!"

Extend Calculus to $\mathbb{C}$ $\quad z = x+iy$

$f : \mathbb{C} \to \mathbb{C} \qquad w = f(z)$

$z = x+iy$, $w = u+iv \qquad u+iv = f(x+iy)$

"$(u,v)$ are determined by $(x,y)$" $\Rightarrow$ View in $\mathbb{R}^2$

$(u,v) = \vec{f}(x,y)$

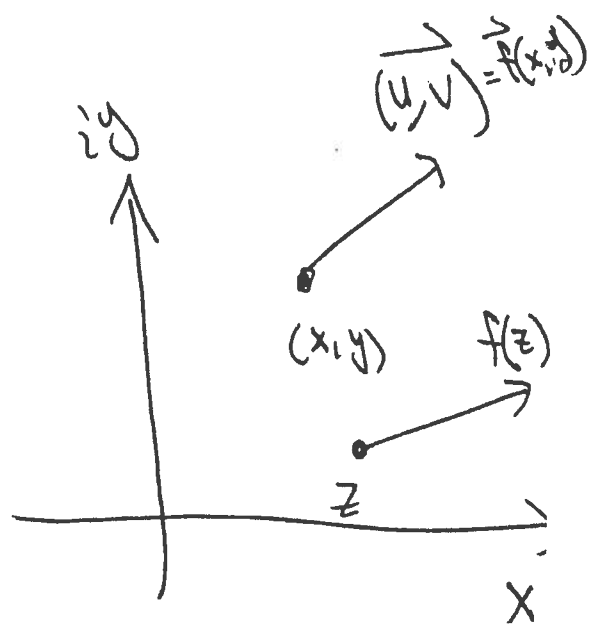
$$\boxed{\big(u(x,y), v(x,y)\big) \equiv \vec{f}(x,y)}$$

But $(u,v)$ should be a vector, so we use notation $f(z) = u+iv$, $\vec{f}(x,y) = \overrightarrow{(u,v)}$

Clearly: a complex valued fn defines a vector field $\overrightarrow{(u(x,y), v(x,y))}$ on $\mathbb{R}^2$

$$\vec{f} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x,y) \longmapsto \overrightarrow{(u,v)}$$

Visualize as Vector Field:



- **Derivative:** $w = f(z)$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$ ← makes sense $\forall \Delta z$ since we can subtr, divide & take limits

$\Delta z = \Delta x + i\Delta y$

<u>Problem</u>: The limit could depend on how we let $\Delta z \to 0$. (There could be a diff limit depending on how we let $\Delta x$ & $\Delta y \to 0$)

<u>Eg</u>: Choose $\Delta z = \Delta x$ <u>real</u>

$$f'(z) = \lim_{\Delta z \to 0} \left[ \frac{f(z+\Delta x) - f(z)}{\Delta x} \right] = \frac{u(x+\Delta x, y) + i v(x+\Delta x, y) - u(x,y) - i v(x,y)}{\Delta x}$$

$$= \lim_{\Delta z \to 0} \left[ \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x} \right]$$

$$= u_x + i v_x$$

- On the other hand — Choose $\Delta z = i\Delta y$ pure imag

$$f'(z) = \lim_{\Delta z \to 0} \left[ \frac{f(z+i\Delta y) - f(z)}{i\Delta y} \right]$$

$$= \lim_{\Delta z \to 0} \left[ \frac{u(x, y+\Delta y) + iv(x, y+\Delta y) - (u(x,y) + iv(x,y))}{i\Delta y} \right]$$

$$= \lim_{\Delta y \to 0} \left[ \frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + \cancel{i} \frac{v(x, y+\Delta y) - v(x,y)}{\cancel{i}\Delta y} \right]$$

$$= v_y - i\, u_{\cancel{x}y}$$

<u>Conclude</u>: in order for both limits to be same we need:

$$\boxed{\begin{aligned} u_x &= v_y \\ v_x &= -u_y \end{aligned}} \qquad (CR)$$

These are called the Cauchy Riemann Eqns

(Assume u,v in C^1, so u_x,u_y,v_x,v_y are continuous)



**Theorem**: $f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$ exists

and is the same for all $\Delta z \to 0$ iff (CR)

**Note**: (CR) is a condition on f !

**Proof**: (Take arbitrary $\Delta z \to 0$, break limit up into $\Delta x$ & $i\Delta y$ pieces & apply CR)
($\Rightarrow$ we just proved; $\Leftarrow$ we return to proof)

**Ex**: $f(z) = z^2$ Show $f'(z) = 2z$

$$f(z) = z^2 = (x+iy)^2 = \underbrace{x^2 - y^2}_{u(x,y)} + \underbrace{2xy}_{v(x,y)}\, i$$

$u_x = 2x = v_y$ ✓

$u_y = -2y = -v_x$ ✓ $\Rightarrow f'(z)$ exists by CR

$$f'(z) = \lim_{\Delta z \to 0}\left[\frac{(z+\Delta z)^2 - z^2}{\Delta z} = \frac{\cancel{z^2} + 2z\Delta z + \Delta z^2 - \cancel{z^2}}{\Delta z}\right]$$

$$= \lim_{\Delta z \to 0} \frac{2z\cancel{\Delta z} + \Delta z^{\cancel{2}}}{\cancel{\Delta z}} = 2z$$

Works same as $\mathbb{R}$ because same algebra = field !

Ex: $f(z) = z^n \quad f'(z) = nz^{n-1}$ FIP

So z^n has anti-deriv 1/(n+1)z^{n+1} for n\neq-1

Same algebra as the real case!

Ex: $f(z) = \frac{1}{z}$ (most important example)

$$f(z) = \frac{1}{z} = \frac{1}{x+iy}\,\frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2} = \underbrace{\frac{x}{x^2+y^2}}_{u} - \underbrace{\frac{y}{x^2+y^2}}_{v}\, i$$

Recall: $r = \sqrt{x^2+y^2}$, $\frac{\partial r}{\partial x} = \frac{x}{r}$, $\frac{\partial r}{\partial y} = \frac{y}{r}$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\frac{x}{r^2} = \frac{1}{r^2} - 2\frac{x}{r^3}\frac{\partial r}{\partial x} = \frac{1}{r^2} - 2\frac{x}{r^3}\frac{x}{r}$$

$$= \frac{1}{r^2} - 2\frac{x^2}{r^4}$$

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{y}{r^2}\right) = -\frac{1}{r^2} + 2\frac{y^2}{r^4}$$

$$\frac{1}{r^2} - 2\frac{x^2}{r^4} \overset{?}{=} -\frac{1}{r^2} + 2\frac{y^2}{r^4}$$

$$\frac{2}{r^2} = 2\frac{x^2+y^2}{r^4} \checkmark$$

Also: $u_y = -\frac{xy}{r^4} = -V_x \checkmark$

- Or just use the same algebra:

$$f'(z) = \lim_{\Delta z \to 0} \left[ \frac{1}{\Delta z}\left(\frac{1}{z+\Delta z} - \frac{1}{z}\right) = \frac{1}{\Delta z}\frac{\cancel{z} - \cancel{z} - \Delta z}{(z+\Delta z) z} = \frac{-1}{(z+\Delta z) z} \right] = -\frac{1}{z^2}$$

Proof shows limit is indept of $\Delta z \to 0 \Rightarrow f'(z)$ exists

- By same algebra as real case –

$n<0$ $\quad f'(z^{-n}) = -n z^{-n-1}$ ~~$n \neq 0$~~ FIP

So: $\frac{d}{dx}(x^n) = n x^{n-1}$
$n \neq 0$
gives anti-derivat
for all $f(z) = z^n$
except $n = -1$

$F(z) = \frac{z^{n+1}}{n+1}$

$n \neq -1$

$F'(z) = z^n$

Note: $f(z) = \frac{1}{z}$ is special!

What about the integral of $f(z)$?
And how do you use it to get
anti-derivative

☑ <u>Integral</u>: We made sense of $f'(z)$, how do we make sense of $\int f(z)\,dz$ ?

<u>Recall</u>: $w = f(z)$ $\quad f: \mathbb{C} \to \mathbb{C}$ $\qquad w = u + iv$, $z = x + iy$

$\Leftrightarrow$ $\overrightarrow{(u,v)} = \vec{f}(x,y)$ $\qquad \vec{r}(t) = \overrightarrow{(x(t),y(t))}$

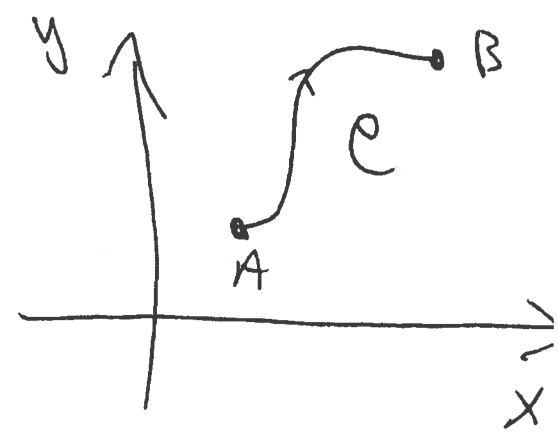
$\qquad\qquad\qquad\qquad\qquad\qquad \vec{G}(x,y) = \overrightarrow{(u(x,y),v(x,y))}$

$\Leftrightarrow$ $u = u(x,y)$, $v = v(x,y)$

$\Leftrightarrow$ $\vec{f}$ is a vector field on $\mathbb{R}^2$.

$\Rightarrow$ Line integrals of vector fields make sense

$\therefore$ Define the integral of $f$ along a curve:




$$\int_C f(z)\,dz = \int_C (u+iv)(dx + i\,dy)$$

$$= \underbrace{\int_C u\,dx - v\,dy}_{\text{Line integral}} + i \underbrace{\int_C v\,dx + u\,dy}_{\text{Line integral}}$$

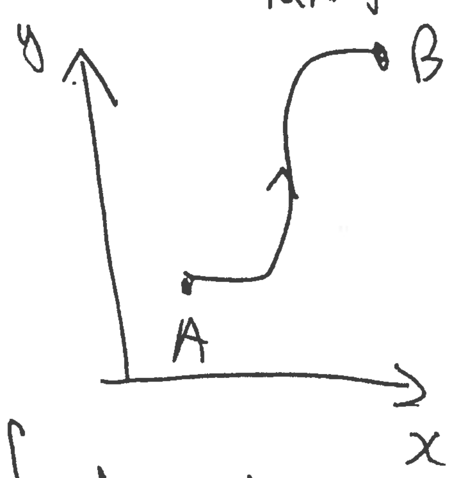
**Defn**: Given a curve $\mathcal{C}$ in xy-plane taking $A=(x_0,y_0)$ to $B=(x_1,y_1)$, define

$$\int_{\mathcal{C}} f(z)\,dz = \underbrace{\int_{\mathcal{C}} u\,dx - v\,dy}_{\text{real \#}} + i \underbrace{\int_{\mathcal{C}} v\,dx + u\,dy}_{\text{real \#}}$$

Recall line integrals in xy-plane $\mathbb{R}^2$: $\mathcal{C}$ a curve taking $A \to B$

Given vector field $\vec{G} = (M,N)$

4 ways to write line integral:




$$\underset{①}{\int_{\mathcal{C}} \vec{G}\cdot\vec{T}\,ds} = \underset{②}{\int_{\mathcal{C}} \vec{G}\cdot d\vec{r}} = \underset{③}{\int_{t_A}^{t_B} \vec{G}\cdot\vec{v}\,dt} = \underset{④}{\int_{\mathcal{C}} M\,dx + N\,dy}$$

① tells what it means

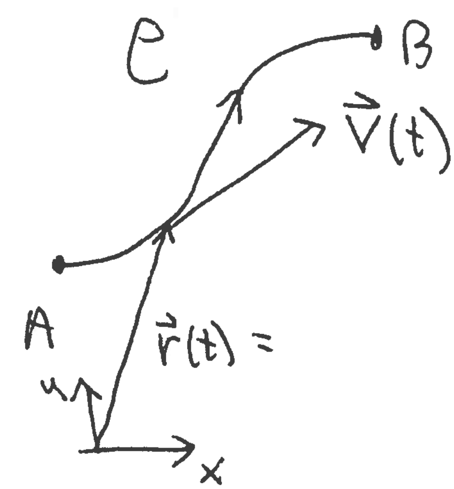
② $\frac{d\vec{r}}{ds} = \vec{T}$ the unit tangent vector

③ tells how to compute it $\vec{v} = \frac{d\vec{r}}{dt} = \frac{ds}{dt}\vec{T}$

④ In terms of a 1-form $M\,dx + N\,dy$

**Ex:** Use Liebniz differential notation to show that all 4 mean the same thing:

**Soln:** parameterize $\mathcal{C}$ wrt parameter $t$




$$x = x(t), \quad y = y(t)$$

$$\mathcal{C} : \vec{r}(t) = \overrightarrow{(x(t), y(t))} \quad t_A \le t \le t_B$$

or $\quad z(t) = x(t) + i\,y(t)$ using complex notation

$\quad z(t_A) = A, \; z(t_B) = B$

**Recall:** $$\vec{v}(t) = \frac{d\vec{r}}{dt} = \underbrace{\frac{ds}{dt}}_{\text{speed}} \underbrace{\frac{d\vec{r}}{ds}}_{\text{unit tangent } \vec{T}} = \overrightarrow{(x'(t), y'(t))}$$

$$\int_{t_A}^{t_B} \vec{G} \cdot \vec{v}\, dt = \int_{t_A}^{t_B} \overrightarrow{(M(\vec{r}(t)), N(\vec{r}(t)))} \cdot \overrightarrow{(x'(t), y'(t))}\, dt$$

$$= \underbrace{\int_{t_A}^{t_B} u(x(t), y(t))\, x'(t) + v(x(t), y(t))\, y'(t)\, dt}_{\text{comput as a 1-D integral}}$$

$$\int_{t_A}^{t_B} \vec{G}\cdot\underbrace{\underbrace{\vec{V}\,dt}_{d\vec{r}}}_{\overrightarrow{(dx,dy)}} = \int_{e} \vec{G}\cdot d\vec{r} = \int_{e} \vec{G}\cdot\vec{T}\,ds$$

$\vec{V} = \frac{d\vec{r}}{dt}$ $\quad d\vec{r} = \vec{v}\,dt$

$\frac{d\vec{r}}{ds} = \vec{T}$ $\quad d\vec{r} = \vec{T}\,ds$

$\vec{V} = \frac{ds}{dt}\vec{T}$ ← unit tangent

↑ vel ↑ speed

$\Rightarrow \vec{v} = \underbrace{\frac{d\vec{r}}{ds}}_{\vec{T}}\frac{ds}{dt}$

$$= \int_{e} \overrightarrow{(M,N)}\cdot\overrightarrow{(dx,dy)}$$

$\vec{V} = \overrightarrow{\left(\frac{dx}{dt}, \frac{dy}{dt}\right)}$

$\vec{v}\,dt = \overrightarrow{(dx,dy)}$

$$= \int_{e} M\,dx + N\,dy$$

✓

<u>Conclude</u> ① ②, ③, ④ give the line integral independent of parameterization, but to compute it you take any parameterizat[ion] $\overrightarrow{(x(t),y(t))}$ and use $dx = x'(t)\,dt$, $dy = y'(t)\,dt$

!

Conclude ② To compute

$$\int_{\mathcal{C}} f(z)\,dz = \int_{\mathcal{C}} u\,dx - v\,dy + i\int_{\mathcal{C}} v\,dx + u\,dy$$

- Give a parameterization of $\mathcal{C}$:

$$z(t) = x(t) + i y(t) \qquad t_A \leq t \leq t_B$$

so $z(t_A) = A$, $z(t_B) = B$

- Compute: $dx = x'(t)\,dt \quad dy = y'(t)\,dt$

(or equivalently dz=z`(t)dt=x`(t)dt+iy`(t)dt)

- Plug in:

$$\int_{\mathcal{C}} u\,dx - v\,dy = \int_{t_A}^{t_B} [u\,x'(t) - v\,y'(t)]\,dt$$

21A integrals

$$\int_{\mathcal{C}} v\,dx + u\,dy = \int_{t_A}^{t_B} [v\,x'(t) + u\,y'(t)]\,dt$$

Q: When is the line integral <u>independent of path</u>?

Ans: When $\vec{G}$ is conservative

<u>Defn.</u> $\vec{G}$ is conservative if $\vec{G} = \nabla g$

for some scalar $g$ $\quad \vec{G} = \overrightarrow{(M,N)} = \overrightarrow{\left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right)}$

In this case –

$$\int_C \vec{G}\cdot\vec{T}\,ds = \int_{t_A}^{t_B} \vec{G}\cdot\vec{v}\,dt = \int_{t_A}^{t_B} \nabla g\cdot\vec{v}\,dt$$

$$= \int_{t_A}^{t_B} \overrightarrow{\left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right)}\cdot\overrightarrow{\left(\frac{dx}{dt}, \frac{dy}{dt}\right)}dt = \int_{t_A}^{t_B} \underbrace{\frac{\partial g}{\partial x}\frac{dx}{dt} + \frac{\partial g}{\partial y}\frac{dy}{dt}}_{\frac{d}{dt} g(\vec{r}(t))}$$

$$= \int_{t_A}^{t_B} \frac{dg}{dt}(\vec{r}(t))\,dt = g(\vec{r}(t_A)) - g(\vec{r}(t_B)) = g(B) - g(A) \checkmark$$

**Theorem**: If $\vec{G} = \overrightarrow{(M,N)} = \nabla g$, then

$$\int_C M\,dx + N\,dy = g(B) - g(A)$$

**Conclude**: the integral is independent of path !


$$\int_{C_1} M\,dx + N\,dy = \int_{C_2} M\,dx + N\,dy = g(B) - g(A)$$

**Cor** ① If $\vec{G} = \nabla g$, then $\oint_C \vec{G}\cdot\vec{T}\,ds = 0$

(Integral around closed paths are zero)

(Pf. $B = A$ !)

**Cor** ② $\vec{G} = \nabla g \Rightarrow g(x,y) = \int_A^{(x,y)} \vec{G}\cdot\vec{T}\,ds + \cancel{\text{const}}\ g(A)$

Pf 210 This gives a general formula for $g$ in terms of $\vec{G}$ when $\vec{G}$ is conservative...

Q: Given $\vec{G}$, how do we know it is conservative? Eg, how do we know $\exists g$ st $\vec{G} = \nabla g$?

<u>Ans</u>: If $\vec{G} = \overrightarrow{(M,N)} = \nabla g = \overrightarrow{(g_x, g_y)}$ then

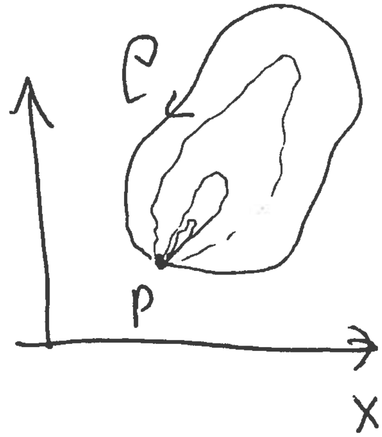
$$\operatorname{Curl} \vec{G} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial_x & \partial_y & \partial_z \\ M & N & 0 \end{vmatrix} = \underset{\sim}{k}\left(N_x - M_y\right) = 0$$

<u>Check</u>: $N_x - M_y = (g_y)_x - (g_x)_y = g_{yx} - g_{xy} = 0.$

<u>Main Thm</u>: it goes the other way, but you have to make sure $\vec{G}$ has no interior singularities where $\operatorname{Curl} \vec{G} = 0$ (like $\vec{G} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$)

<u>Thm</u>: If $\operatorname{Curl} \vec{G} = 0$ in a simply connected region $D \subseteq \mathbb{R}^2$, then $\exists\, g$ st $\vec{G} = \nabla g$ in $D$

<u>Defn</u>: D is simply connected if every closed curve in D can be contracted continuously to a point w/o the curve passing thru a pt not in D.




This excludes point singularity like $\frac{(-y, x)}{x^2+y^2} = \vec{G}$: Not defined so

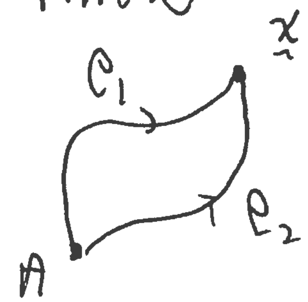
$\text{Curl}\,\vec{G} \neq 0$ at $x = y = 0$.

<u>Proof</u>. Recall Greens Thm:

$$\oint_C \vec{G} \cdot \vec{T}\, ds = \oint_C M dx + N dy = \iint_A \underbrace{N_x - M_y}_{\text{Curl}\,\vec{G}}\, dA$$

where C is a closed curve in D & A is the area surrounded by C. Since D is simply connected, $\text{Curl}\,\vec{G} = 0 \Rightarrow \text{RHS} = 0$
$\Rightarrow \text{LHS} = 0 \;\forall$ closed curve C in D.

Since $\oint_{C} \vec{G}\cdot\vec{T}\,ds = 0$, we know




$$g(\underset{\sim}{x}) = \int_{A}^{\underset{\sim}{x}} \vec{G}\cdot\vec{T}\,ds$$

is defined independent of path from A to $\underset{\sim}{x} = (x,y) \in D$. Turns out (21D)

$$\nabla g = \vec{G} \qquad \left(0 = \int_{C_1 - C_2} = \int_{C_1} - \int_{C_2} \checkmark\right)$$

so $\vec{G}$ is conservative. ✓

<u>Note</u>: we need $\operatorname{curl}\vec{G} = 0$ at <u>every</u> point of A inside C!

2 Steps to discover complex FTC from Mat21D

- Formally: Let $f(z) = u(x,y) + i v(x,y)$. Then

$$\int_{e} f(z)\,dz = \int_{e} (u+iv)(dx+i\,dy) = \underbrace{\int_{e} u\,dx - v\,dy}_{\int_{e} \vec{G}_1 \cdot \vec{T}\,ds} + i \underbrace{\int_{e} v\,dx + u\,dy}_{\int_{e} \vec{G}_2 \cdot \vec{T}\,ds}$$

The real and complex parts are real 21D Line integrals !

$$\vec{G}_1 = \overrightarrow{(u,-v)} \qquad \vec{G}_2 = \overrightarrow{(v,u)}$$

- Recall: 21D – If $\operatorname{Curl} \vec{G} = 0 = N_x - M_y$ then $\exists$ $g(x,y)$ such that $\vec{G} = (M,N) = \nabla g$.

- Recall: 21D – $\operatorname{Curl} \vec{G}$ must vanish in a simply connected domain, meaning no holes where $\operatorname{Curl} \vec{G}$ is undefined

"Proof" ~~Stokes~~ Green's $\displaystyle\int_{e} \vec{G} \cdot \vec{T}\,ds = \iint_{A} \underbrace{N_x - M_y}_{\operatorname{Curl} \vec{G}}\,dA = 0$ $\forall$ closed Curve $e$

- Since $\vec{G}_1$ & $\vec{G}_2$ involve $u, v$, lets <u>assume</u> $f(z)$ is analytic so C-R holds -

$$u_x = v_y \quad , \quad u_y = -v_x$$

$$u_x - v_y = 0 \qquad u_y + v_x = 0$$

Then $\operatorname{Curl}\vec{G}_1 = \operatorname{Curl}(\overrightarrow{u,-v}) = (-v)_x - (u)_y = 0$

$(N_x - M_y)$

$\operatorname{Curl}\vec{G}_2 = \operatorname{Curl}(\overrightarrow{v,u}) = (u)_x - (v)_y = 0$

<u>Conclude</u>: C-R $\Rightarrow$ $G_1$ & $G_2$ are Curl Free!

~~[illegible]~~

- $\operatorname{Curl} G_i = 0 \Rightarrow \exists\ U(x,y)$ st $\nabla U = \vec{G}_1$
  $V(x,y)$ st $\nabla V = \vec{G}_2$

$\therefore \int_e \vec{G}_1 \cdot \vec{T}\, ds = \int_e \nabla U \cdot \vec{T}\, ds = \int_{t_A}^{t_B} \frac{d}{dt} U(x(t), y(t))\, dt = U(B) - U(A)$

$\int_e \vec{G}_2 \cdot \vec{T}\, ds = V(B) - V(A)$

<u>Conclude</u>: If $f(z) = u + iv$ satisfies C-R eqns then

$$\int_{\mathcal{C}} f(z)\,dz = \left[U(B) - U(A)\right] + i\left[V(B) - V(A)\right]$$

$$= \left[U(B) + iV(B)\right] - \left[U(A) + iV(A)\right]$$

$$= F(B) - F(A)$$

$$F(z) = U(z) + iV(z)$$

- To complete complex FTC, we only need

$$F'(z) = f(z)$$

But we have: $\nabla U = \vec{G}_1 = \overrightarrow{(u, -v)}$

$$\nabla V = \vec{G}_2 = \overrightarrow{(v, u)}$$

so $U_x = u$, $U_y = -v$
$V_x = v$, $V_y = u$ $\Rightarrow U_x = V_y,\ U_y = -V_x$

$\Rightarrow F$ satisfies C-R Equations! $\Rightarrow F'(z)$ <u>exists</u>

But by Defn –

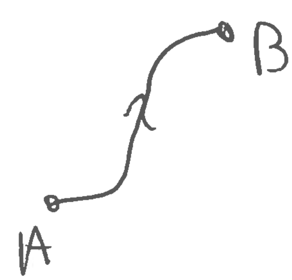
$$F'(z) = \lim_{\Delta x \to 0} \frac{F(x+\Delta x + iy) - F(x+iy)}{\Delta x}$$

$$= U_x + i V_x = u + iv = f(z) \checkmark$$

We have complex FTC:

<u>Thm</u>: (FTC). Assume $f$ satisfies C-R Equn.

Then $\int_C f(z)\,dz$ is indept of path from A to B




Moreover, $\exists$ $F(z)$ such that $F'(z)$ exists, and

$$\int_C f(z)\,dz = F(B) - F(A)$$

$$F'(z) = f(z)$$

Pretty amazing!

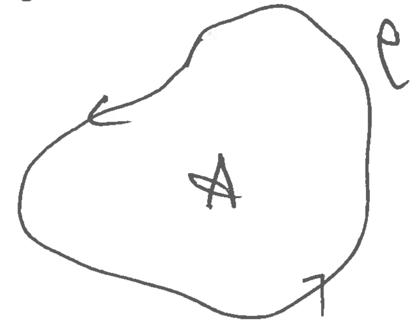
☐ Our key result from 21D is

**Thm**: If $\text{Curl}\,\vec{G} = 0$ (in a s.c. region), then $\exists\, g$ st $\nabla g = \vec{G}$. Lets recall the proof

- Recall: $\text{Curl}\,\vec{G} = 0 \Rightarrow$ by Green's Thm

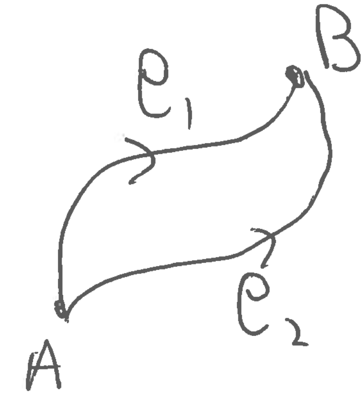
$$\oint_{\mathcal{C}} \vec{G}\cdot\vec{T}\,ds = \iint_{A} \underbrace{\cancelto{0}{N_x - M_y}}_{\text{Curl}\,\vec{G}}\,dA = 0$$

any simple closed curve bounding area $A$




$\Rightarrow$ integral around closed curves $= 0$

$\Rightarrow$ integral is indept of path




$$\int_{\mathcal{C}_1 - \mathcal{C}_2} \vec{G}\cdot\vec{T}\,ds = \int_{\mathcal{C}_1} \vec{G}\cdot\vec{T}\,ds - \int_{\mathcal{C}_2} \vec{G}\cdot\vec{T}\,ds = 0$$

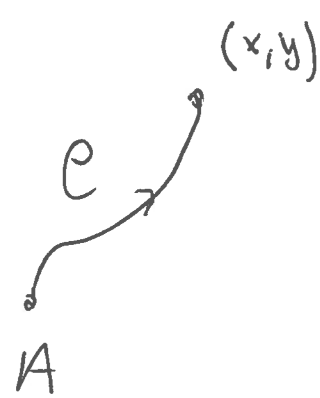
closed curve

line integral is linear on paths

Thus fixing A define

$$g(x,y) = \int_A^{(x,y)} \vec{G} \cdot \vec{T}\, ds$$

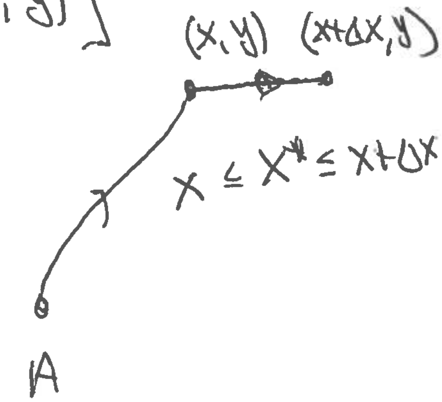
$\uparrow$ indept of path from A to $(x,y)$




Claim: $\nabla g = \vec{G} = \overrightarrow{(M,N)}$ $\quad \big(\nabla g_1 = \nabla g_2 \Rightarrow \nabla(g_2 - g_1) = 0$

$\Rightarrow g_2 - g_1 = \text{const}\big)$

Pf. $\dfrac{\partial g}{\partial x} = \lim\limits_{\Delta x \to 0} \dfrac{1}{\Delta x}\left[g(x+\Delta x, y) - g(x,y)\right]$




Since g defined indept of path, take path to $(x+\Delta x, y)$ to be path to $(x,y)$ + path along $\parallel$ x-axis.

$$\frac{\partial g}{\partial x} = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int_{(x,y)}^{(x+\Delta x, y)} \vec{G} \cdot \vec{T}\, ds \qquad \vec{r}(t) = \overrightarrow{(x+t, y)}$$

$$\vec{r}\,'(t) = (1,0)$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\int_0^{\Delta x} \vec{G} \cdot \vec{r}\,'(t)\, dt = \int_0^{\Delta x} M(x+t, y)\, dt\right] \overset{\lim}{=} M(x^*, y)$$

$\uparrow$ Mean Value Thm

Continuous function of t

$= M(x,y)$ ✓

4 Summary - we have proven:

<u>Theorem</u>: If $f$ satisfies C-R equations (meaning $u_x, v_x, u_y, v_y$ exist, $u_x = v_y$ & $v_x = -u_y$, so $f'(z)$ exists indept of $\Delta z \to 0$) in a s.c. region $D \subseteq \mathbb{C}$, then there exists $F(z) = U(x,y) + iV(x,y)$ s.t.

$$F'(z) = f(z)$$

and

$$\int_{\mathcal{C}} f(z)\,dz = F(B) - F(A) \qquad \text{(FTC)}$$

<u>Note</u>: if $F'(z) = H'(z)$ then $F(z) = H(z) + \text{const}$ (FIP)

As a Corollary we obtain Cauchy's Thm

<u>Theorem</u>: (Cauchy) If $f'(z)$ exists in a s.c. domain $D \subseteq \mathbb{C}$, then $\oint_{\mathcal{C}} f(z)\,dz = 0$.

Pf $A = B \Rightarrow \oint_{\mathcal{C}} f(z)\,dz = \underbrace{F(B) - F(A)}_{A=B} = 0$

The next theorem says that all you need is an anti-derivative of $f(z)$ in a simply conneted Domain $D \subseteq \mathbb{R}^2$ to conclude the FTC

**Theorem:** Assume $F'(z) = f(z)$, and f cont in a ~~simply connect~~ domain $D \subseteq \mathbb{R}^2$ (not nec s.c.), where $f(z) = u + iv$ & $u(x,y), v(x,y)$ are differentiable.
then $f$ is differentiable and

$$\int_{C} f(z)\,dz = F(B) - F(A).$$

**Proof:** Since $F$ is differentiable, the CR eqns hold: $F(z) = U + iV$, $U_x = V_y$, $U_y = -V_x$.
But $F'(z) = f(z) = u + iv \Rightarrow U_x = u$, $V_x = v$

Thus
$$\left.\begin{aligned} u_x &= U_{xx} = V_{yx} = V_{xy} = v_y \\ u_y &= U_{xy} = U_{yx} = -V_{xx} = -v_x \end{aligned}\right\} \text{C-R for } f \Rightarrow f \text{ diff}$$

Thus $f$ is differentiable.

(At this point we could apply the first theorem if we assumed s.c.…)

Claim: $\vec{G}_1, \vec{G}_2$ conservative because $\nabla U = \vec{G}_1, \nabla V = \vec{G}_2$

I.e., $F'(z) = f(z) = u + iv$ so

$(\Delta z = \Delta x)$ $\quad U_x + iV_x = u + iv \quad U_x = u,\ V_x = v$

$(\Delta z = i\Delta y)$ $\quad V_y - iU_y = u + iv \quad U_y = -v,\ V_y = u$

$$\Rightarrow \quad \nabla U = \overrightarrow{(U_x, U_y)} = \overrightarrow{(u, -v)} = \vec{G}_1$$

$$\nabla V = \overrightarrow{(V_x, V_y)} = \overrightarrow{(v, u)} = \vec{G}_2$$

Thus: $$\int_{\mathcal{C}} \vec{G}_1 \cdot \vec{T}\, ds = \int_{t_A}^{t_B} \nabla U \cdot \frac{d\vec{r}}{dt}\, dt$$

$$= \int_{t_A}^{t_B} \frac{d}{dt} U(\vec{r}(t))\, dt = U(B) - U(A)$$

Similarly, $\int_{\mathcal{C}} \vec{G}_2 \cdot \vec{T}\, ds = V(B) - V(A)$

so $\int_{\mathcal{C}} f(z)\, dz = F(B) - F(A)$ ✓

The whole subject of complex variables gets extremely interesting when we consider functions like $f(z) = \frac{1}{z^n}$ which are differentiable for $z \neq 0$, a domain with an interior singularity, not simply connected.

We now show that $f(z) = \frac{1}{z}$ is the fundamental singularity of complex variables. We show:

$$\oint_{\mathcal{C}} z^n \, dz = \begin{cases} 0 & \text{if } n \neq -1 \\ 2\pi i & \text{if } n = -1 \end{cases}$$

$n \in \mathbb{Z} = \{\ldots -3, -2, -1, 0, 1, 2, 3 \ldots\}$

First
~~To finish~~, we prove

$$\int_{\mathcal{C}} z^n \, dz = 0 \quad \text{for } n \neq -1.$$

But in this case, $\frac{d}{dz}\left(\frac{z^{n+1}}{n+1}\right) = z^n$

so $F(z) = \frac{z^{n+1}}{n+1} \Rightarrow F'(z) = z^n \quad z \neq 0.$

Thus (we don't need s.c. domain)

$$\int_{\mathcal{C}} z^n \, dz = F(B) - F(A) = 0 \quad B = A.$$

✓

$$\oint_{\mathcal{C}} \frac{dz}{z} = 2\pi i \neq 0 \,!$$

Thm:

Conclude: $f(z) = \frac{1}{z}$ is fundamental to Complex variables, & its anti-derivati

$F(z) = \log z$ cannot be defined for $z \neq 0$

- or else we'd have $\oint_{\mathcal{C}} \frac{dz}{z} = 0$ !

First, let $\mathcal{C}_0$ denote the special curve which is the unit circle centered at $z=0$, with parameterization $z(t)=x(t)+iy(t)$

$$\begin{aligned} x(t) &= \cos t \\ y(t) &= \sin t \end{aligned} \quad t_A \le t \le t_B \quad t_A=0,\ t_B=2\pi$$

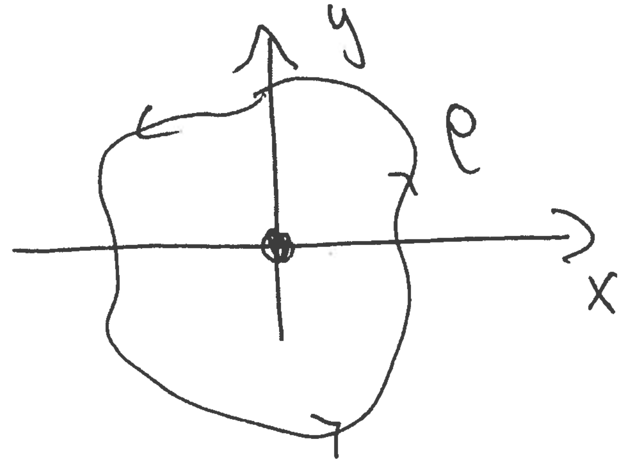
We first prove $\displaystyle\int_{\mathcal{C}} \frac{dz}{z} = \int_{\mathcal{C}_0} \frac{dz}{z}$.

$$\int_{\mathcal{C}} \frac{dz}{z} = \int_{\mathcal{C}} \frac{dx+i\,dy}{z}\cdot\frac{\bar z}{\bar z} = \int_{\mathcal{C}} \frac{(dx+i\,dy)(x-iy)}{x^2+y^2}$$

$$= \underbrace{\int_{\mathcal{C}} \frac{x\,dx+y\,dy}{x^2+y^2}}_{\vec G_1\cdot\vec T} + i\underbrace{\int_{\mathcal{C}} \frac{x\,dy-y\,dx}{x^2+y^2}}_{\vec G_2\cdot\vec T}$$

We know $\frac{d}{dz}\left(\frac{1}{z}\right) = -\frac{1}{z^2}$ so $\operatorname{Curl}\vec G_1 = 0 = \operatorname{Curl}\vec G_2$ for $z\neq 0$

Thm: Let $\mathcal{C}$ be any closed curve surrounding $z=0$.



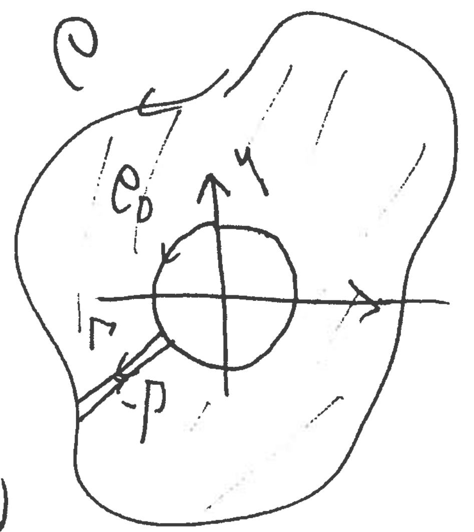

Then

$$\oint_{\mathcal{C}} \frac{dz}{z} = 2\pi i$$

Note: $f(z) = \frac{1}{z} \Rightarrow f'(z) = -\frac{1}{z^2}$ for $z \neq 0$, so $f'(z)$ exists but not in a simply connected domain. Thus Cauchy's Thm does not apply.

We use defn of line integral to calculate $\oint_{\mathcal{C}} \frac{dz}{z}$ ...

but $\vec{G}_1$ & $\vec{G}_2$ are not Curl-free in a simply connected region. So first — $e_0 + \Gamma - e - \Gamma$ is a ~~sim~~ a closed curve surroundy the simply connected domain betw $e$ & $e_0$ (excluding $\Gamma$). Thus since $\vec{G}_1$ & $\vec{G}_2$ are Curl-free in this s.c. domain, we know




$$0 = \int_{e_0+\Gamma-e-\Gamma} \vec{G}_i \cdot \vec{T}\, ds = \int_{e_0} + \cancel{\int_{\Gamma}} - \int_{e} - \cancel{\int_{\Gamma}}$$

$$\Rightarrow \int_{e_0} \vec{G}_i \cdot \vec{T} = \int_{e} \vec{G}_i \cdot \vec{T} \qquad i = 1,2$$

$$\therefore \int_{e} f(z)\,dz = \int_{e_0} f(z)\,dz \quad \text{with } f(z) = \frac{1}{z}.$$

Now we compute $\int_{C_0} \frac{dz}{z}$ :

$$\int_{C_0} \frac{dz}{z} = \int_{C_0} \frac{x\,dx + y\,dy}{x^2+y^2} + i \int_{C_0} \frac{x\,dy - y\,dx}{x^2+y^2}$$

$x = x(t) = \cos t \qquad dx = -\sin t\, dt$

$y = y(t) = \sin t \qquad dy = \cos t\, dt \qquad 0 \le \theta \le 2\pi$

$$= \int_0^{2\pi} \frac{(\cos t)(-\sin t) + (\sin t)(\cos t)}{1}\, dt$$

$$+ i \int_0^{2\pi} \frac{(\cos t)(\cos t) + (-\sin t)(-\sin t)}{1}\, dt$$

$= 2\pi i$ ✓