II Proof of C-R Theorem:

☐ The proof of the C-R Theorem the hard way.

<u>Thm</u> (C-R) Assume $f(z) = u(x,y) + iv(x,y)$, and assume u and v are differentiable functions (in the sense of <u>real</u> valued $u: \mathbb{R}^2 \to \mathbb{R}$, $v: \mathbb{R}^2 \to \mathbb{R}$) at $z_0 = x_0 + iy_0$. Then

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

exists independent of $\Delta z \to 0$ iff CR holds:

$$u_x = v_y$$
$$u_y = -v_x$$

Pf. ($\Rightarrow$) We did this by taking $\Delta z = \Delta x$ & $\Delta z = i\Delta y$

($\Leftarrow$) This is the hard way.

To start use notation $\underset{\sim}{x} = (x,y)$, $\underset{\sim}{x}_0 = (x_0, y_0)$

$$|\underset{\sim}{x}| = \sqrt{x^2+y^2} = |z|$$

- We start by recalling the defn of differentiability for real valued functions.

<u>Defn</u> : $u(x,y) = u(\underset{\sim}{x})$ is differentiable at $\underset{\sim}{x}_0 = (x_0, y_0)$ if $\exists$ a vector $\nabla u \equiv \nabla u(\underset{\sim}{x}_0)$ s.t.

$$\frac{u(\underset{\sim}{x}) - u(\underset{\sim}{x}_0)}{|\underset{\sim}{x} - \underset{\sim}{x}_0|} - \nabla u \cdot \frac{\underset{\sim}{x} - \underset{\sim}{x}_0}{|\underset{\sim}{x} - \underset{\sim}{x}_0|} \xrightarrow{\text{as } \underset{\sim}{x} \to \underset{\sim}{x}_0} 0 \qquad (*)$$

For example, assuming $(*)$, if $\underset{\sim}{x} - \underset{\sim}{x}_0 = t\vec{v}$ (i.e., we restrict to line thru $\vec{v}$), then

$$\frac{d}{dt} u(\underset{\sim}{x} + t\vec{v})\Big|_{t=0} = \nabla u \cdot \vec{v} \qquad \left(\begin{array}{c}\text{giving formula for} \\ \text{directional deriv's}\end{array}\right).$$

Taking $\vec{v} = \vec{e}_1 = \overrightarrow{(1,0)}$ then gives $u_x = \nabla u \cdot \vec{e}_1$
$\vec{v} = \vec{e}_2 = \overrightarrow{(0,1)}$ then gives $u_y = \nabla u \cdot \vec{e}_2$

$\Rightarrow \nabla u = (u_x, u_y)$. So $\nabla u = (u_x, u_y)$ follows from $(*)$.

- Now rewrite (*) as

$$u(\underset{\sim}{x}) - u(\underset{\sim}{x}_0) - \nabla u \cdot (\underset{\sim}{x} - \underset{\sim}{x}_0) = o(1)|\underset{\sim}{x} - \underset{\sim}{x}_0|$$

Here "o(1)" is "little oh of one" denotes ~~any~~ some function which tends to zero as $\underset{\sim}{x} \to \underset{\sim}{x}_0$.

Conclude: If both u & v are differentiable at $\underset{\sim}{x} = \underset{\sim}{x}_0$, we have that there exist vectors ∇u_0 & ∇v_0 such that

$$u(\underset{\sim}{x}) - u(\underset{\sim}{x}_0) = \nabla u_0 (\underset{\sim}{x} - \underset{\sim}{x}_0) + o(1)|\underset{\sim}{x} - \underset{\sim}{x}_0|$$

$$v(\underset{\sim}{x}) - v(\underset{\sim}{x}_0) = \nabla v_0 (\underset{\sim}{x} - \underset{\sim}{x}_0) + o(1)|\underset{\sim}{x} - \underset{\sim}{x}_0|$$

or in vector form

$$\begin{pmatrix} u \\ v \end{pmatrix}(\underset{\sim}{x}) - \begin{pmatrix} u \\ v \end{pmatrix}(\underset{\sim}{x}_0) = \underbrace{\begin{bmatrix} -\nabla u_0- \\ -\nabla v_0- \end{bmatrix}}_{\text{Jacobian} = J(\underset{\sim}{x}_0)} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + o(1)|\underset{\sim}{x} - \underset{\sim}{x}_0|$$

Therefore: If $\begin{pmatrix} u \\ v \end{pmatrix}$ is diff @ $\underset{\sim}{x}_0$, then $J = \begin{pmatrix} u_x, & u_y \\ v_x & v_y \end{pmatrix}$

Now $f'(z)$ exists at $z=z_0$ if $\exists\ a+ib \in \mathbb{C}$

s.t. $\displaystyle\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = a+ib$ (**)

in which case $a+ib = f'(z_0)$ by definition.

But (**) is equivalent to

$$f(z)-f(z_0) = (a+ib)(z-z_0) + o(1)|z-z_0|$$

which is equiv to

$|\Delta z| = |\Delta \underline{x}| = \sqrt{\Delta x^2+\Delta y^2}$

$$\begin{pmatrix} u \\ iv \end{pmatrix} - \begin{pmatrix} u_0 \\ iv_0 \end{pmatrix} = \left[(a+ib)(\Delta x + i\Delta y)\right] + o(1)|\Delta z|$$

$$\Downarrow$$

$$\left[a\Delta x - b\Delta y + i(b\Delta x + a\Delta y)\right]$$

$$\Downarrow$$

$$\begin{bmatrix} a & -b \\ bi & ai \end{bmatrix} \begin{bmatrix} \Delta x \\ \cancel{i}\Delta y \end{bmatrix}$$

Conclude: if we can find (a,b) such that

(A) $$\begin{pmatrix} u \\ iv \end{pmatrix} - \begin{pmatrix} u_0 \\ iv_0 \end{pmatrix} - \begin{bmatrix} a & -b \\ bi & ai \end{bmatrix} \begin{bmatrix} \Delta x \\ \cancel{i}\Delta y \end{bmatrix} = o(1)|\Delta \underline{x}|$$

then $f'(z_0)$ exists & equals $a+ib$.

- But by differentiability we know

$$(B)\quad \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} + o(1)|\Delta \underset{\sim}{x}|.$$

So if we assume C-R $u_x = v_y = a$, $u_y = -v_x = -b$, ~~and replace $v \to iv$, $\Delta y \to i\Delta y$~~, this gives

$$(C)\quad \begin{pmatrix} u \\ iv \end{pmatrix} - \begin{pmatrix} u_0 \\ iv_0 \end{pmatrix} = \begin{bmatrix} a & -b \\ bi & ai \end{bmatrix} \begin{pmatrix} \Delta x \\ \cancel{i}\Delta y \end{pmatrix} + o(1)|\Delta z|$$

and from this we conclude (cf (A), (C))

$$f'(z_0) = a + ib.$$

We Have: differentiability of u & v together with C-R $\Rightarrow$ $f'(z_0)$ exists ✓