III complex exponential & logarithm:

- Recall for real #'s:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}. \qquad \sin x = \sum_{k=1} (-1)^{k+1} \frac{x^{2k-1}}{(2k-1)!}.$$

$$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}.$$

Property: $e^{x+y} = e^x e^y$, $e^{kx} = (e^x)^k$

Q: How to extend these to complex functions of $z = x+iy$ so they're analytic?
(We know powers are analytic!)

- <u>Define</u>: $e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{n=0}^{\infty} \frac{(i)^n x^n}{n!}.$

$$e^{ix} = \sum_{n=0}^{\infty} (i)^{2n} \frac{x^{2n}}{(2n)!} + \sum_{n=1}^{\infty} (i)^{2n-1} \frac{x^{2n-1}}{(2n-1)!}.$$

even powers   odd powers

$(i)^{2n} = (i^2)^n = (-1)^n$, $(i)^{2n-1} = \frac{i^{2n}}{i} = -(-1)^n i = (-1)^{n+1} i$

Conclude:

$$e^{ix} = \cos x + i\sin x$$

By this it makes sense to define

$$(*) \quad e^{z} = e^{x+iy} = e^{x} e^{iy} = e^{x}(\cos y + i\sin y)$$

$$e^{z} = \underbrace{e^{x}\cos y}_{u} + i\,\underbrace{e^{x}\sin y}_{v}$$

check - CR equations hold:

$$u_x = e^{x}\cos y = v_y$$

$$u_y = -e^{x}\sin y = -v_x$$

Conclude: e^{z} is analytic $\forall\, z \in \mathbb{C}$

So $(*)$ extends e^{x} to an analytic complex function $e^{z} : \mathbb{C} \to \mathbb{C}$.

2B

Since C-R hold <u>everywhere</u>, we know

$$\frac{d}{dz}e^z = \lim_{\Delta z \to 0} \frac{e^{z+\Delta z} - e^z}{\Delta z}$$

exists indept $\Delta z \to 0$, and

$$\frac{d}{dz}e^z \underset{\Delta z = \Delta x}{\overset{\uparrow}{=}} u_x + i u_x$$

$u = e^x \cos y$

$v = e^x \sin y$

$$u_x = \frac{\partial}{\partial x} e^x \cos y \qquad v_x = \frac{\partial}{\partial x} e^x \sin y$$

$$= u \qquad\qquad\qquad = v$$

$\therefore \frac{d}{dz}e^z = e^z$. Also: $\oint_C e^z \, dz = 0$

Alternatively: if we know we can T×T diff

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad \frac{d}{dz}e^z = \sum_{n=1}^{\infty} \frac{n}{n!} z^{n-1} = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$= e^z$$

- The exponential provides geometric interp of complex multiplicaton:

$$f(z) = e^x e^{iy} = e^x(\cos y + i\sin y)$$

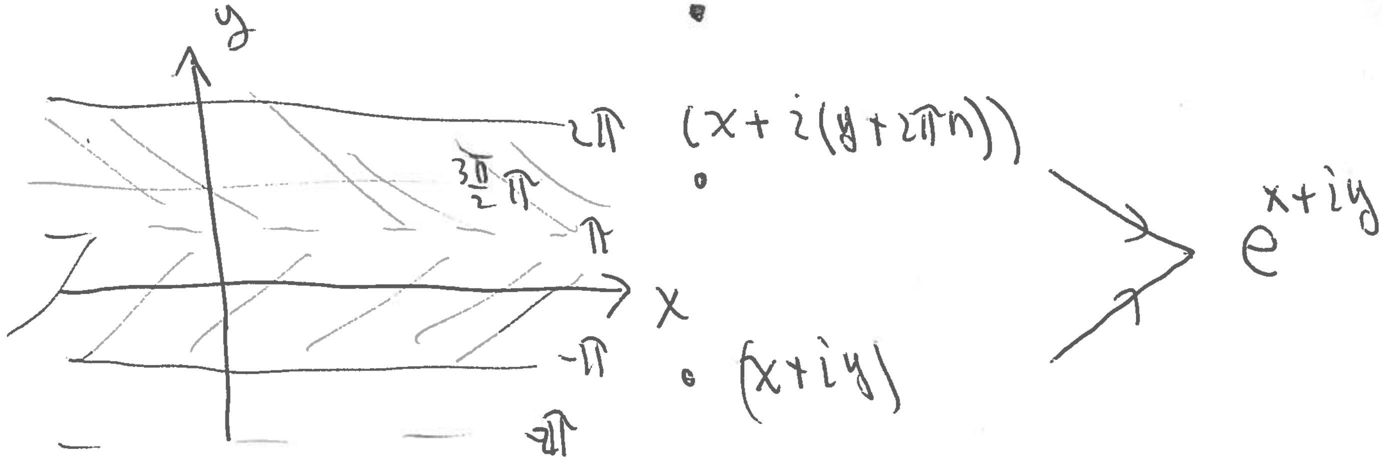
Properties (easy to check from defn)

(i) $e^{z+w} = e^z e^w$ (check – $e^{i\theta_1} + e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$ FP)

(ii) $e^z \neq 0$

(iii) $|e^{x+iy}| = e^x$

(iv) e^z is 2π-periodic in y $\quad e^{x+iy} = e^{x+i(y+2\pi n)}$



Ex: $e^{i\pi} = \cos\pi + i\sin\pi = -1$

$$\boxed{e^{i\pi} + 1 = 0}$$

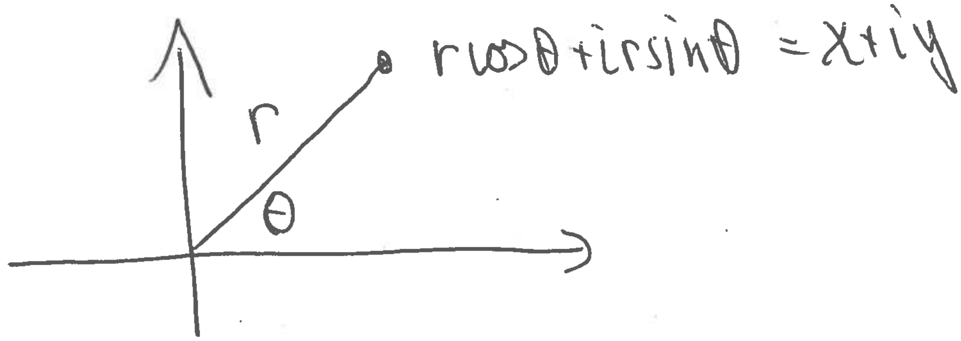
relates all the famous #'s of classical math: $e, i, \pi, 1, 0$

- Writing $z = x + iy$ $\quad |z| = \sqrt{x^2+y^2} = r$

$$\theta = \operatorname{Arg} z \in (-\pi, \pi]$$

$$z = re^{i\theta} = |z|\underbrace{(\cos\theta + i\sin\theta)}_{\overrightarrow{(\cos\theta,\ \sin\theta)}\ \text{unit vector}}$$

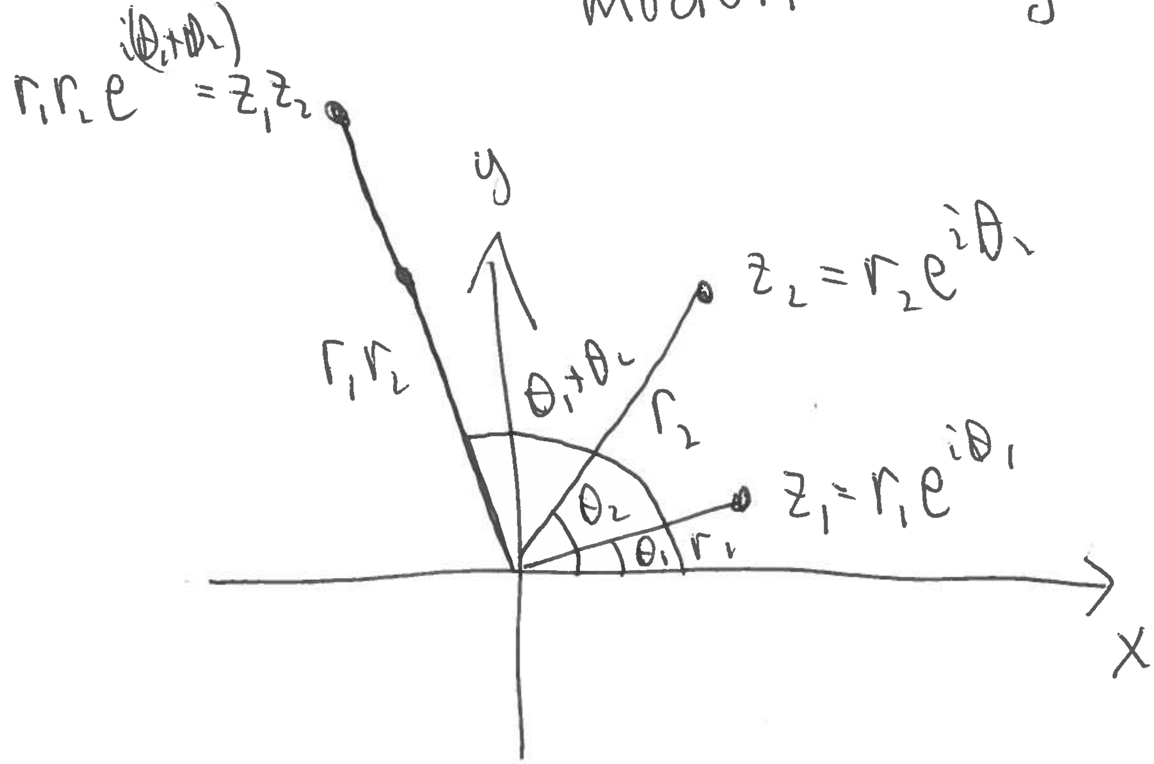
r = modulus



- Writing $z = re^{i\theta}$ gives geom interpretation of complex multiplication —

$$z_1 \cdot z_2 = r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = \underbrace{r_1 r_2}_{\text{mult moduli}} \underbrace{e^{i(\theta_1+\theta_2)}}_{\text{add angles}}$$

Picture



Q: How to extend $\cos x$ and $\sin x$ to $\cos z$ & $\sin z$ for $z = x+iy$?

Start with $e^{ix} = \cos x + i\sin x \qquad x \in \mathbb{R}$

<u>Claim</u>:
$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

<u>Check</u>:
$$\frac{e^{ix} - e^{-ix}}{2i} = \frac{\cancel{\cos x} + i\sin x - \overset{\cancel{\cos x}}{\cos(-x)} - i\overset{-\sin x}{\sin(-x)}}{2i}$$

$$= \frac{\cancel{2}i\sin x}{\cancel{2}i} = \sin x \checkmark$$

Similarly for $\cos x$.

<u>Define</u>: $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i} \qquad \cos z = \dfrac{e^{iz} + e^{-iz}}{2}$

Claim: This extends $\cos x$ and $\sin x$ (7)

to an everywhere analytic fn of z.

Defn: f is entire, i.e. $f'(z)$ exists $\forall z \in \mathbb{C}$

Pf. We show CR equations hold

$$W = \cos z = \frac{e^{iz} + e^{-iz}}{2} = \frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2}$$

$$= \frac{e^{-y}(\cos x + i\sin x) + e^{y}(\cos(-x) + i\sin(-x))}{2}$$

$$= \frac{1}{2}\left(e^{-y}\cos x + e^{y}\cos x\right) + i\left(e^{-y}\sin x - e^{y}\sin x\right)$$

$$= \underbrace{\frac{e^{-y}+e^{y}}{2}\cos x}_{u(x,y)} + i\,\underbrace{\frac{e^{-y}-e^{y}}{2}\sin x}_{v(x,y)}$$

To show $w=\cos z$ is entire, it suffices to show C-R eqn's hold everywhere ($\forall z$)

$$u_x = -\frac{e^{-y}+e^{y}}{2}\sin x, \quad v_y = -\frac{e^{-y}+e^{y}}{2}\sin x$$

$$u_y = \frac{-e^{-y}+e^{y}}{2}\cos x, \quad +v_x = +\frac{e^{-y}-e^{y}}{2}\cos x$$

$$u_x = v_y$$

$$u_y = -v_x$$

Turns out $\exists$ only at most one way to extend a real valued fn to complex analytic fn.

Since $\cos z$ & $\sin z$ satisfy C-R everywhere, we know $\frac{d}{dz}\cos z$ & $\frac{d}{dz}\sin z$ exist.

Calculate:

$$\frac{d}{dz}\cos z = \frac{d}{dz}\frac{e^{iz}+e^{-iz}}{2} = \frac{ie^{iz}-ie^{-iz}}{2}$$

$$= -\frac{e^{iz}-e^{-iz}}{2i} = -\sin z$$

$$\frac{d}{dz}\sin z = \frac{d}{dz}\frac{e^{iz}-e^{-iz}}{2i} = \cos z.$$

we used the Chain Rule

<u>Note</u>: All Alg properties of complex derivatives follow from Field Axioms — f, g diff $\Rightarrow$

① $(f+g)' = f' + g'$; $\left(\frac{f}{g}\right)'_{g\neq 0} = \frac{gf' - fg'}{g^2}$; $(f\cdot g)' = fg' + f'g$

② Chain rule: $\frac{d}{dz} f(g(z)) = f'(g(z))\cdot g'(z)$

---

③ $\cancel{w = f(z)}$ ~~1-1 onto~~ $\cancel{\Rightarrow f^{-1}(w) = z}$ $\cancel{f^{-1}(f(z)) = z}$ $\cancel{\Rightarrow f^{-1\prime}(w)\cdot f'(z) = 1}$ $\cancel{f^{-1\prime}(w) = \frac{1}{f'(z)}}$

3 $w = f(z)$ has an inverse if f is

1-1: $f(z_1) = f(z_2) \Rightarrow z_1 = z_2$

onto: $\forall w \; \exists z$ st $w = f(z)$.

$$\left.\begin{array}{l} f^{-1}(w) = z \\ w = f(z) \end{array}\right\} \; f^{-1}(f(z)) = z$$

Diff both sides: $f^{-1\prime}(w) \cdot f'(z) = 1$

$$f^{-1\prime}(w) = \frac{1}{f'(z)}.$$

Note: Since e^z, $\sin z$, $\cos z$ are entire (Differentiable in all of $\mathbb{C}$) we know

$\oint_C e^z dz = 0$, $\oint_C \cos z \, dz = 0$, $\int_C \sin z \, dz = 0$.

The logarithm:

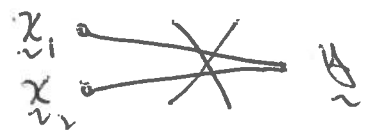
- We construct the inverse of $w = e^z$

Recall: If $F : \mathbb{R}^n \to \mathbb{R}^n$ is 1-1 and onto

1-1 "no two points hit same target"

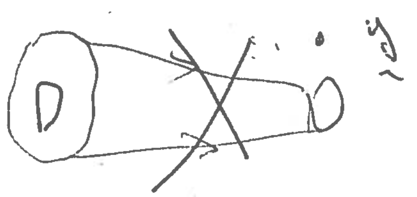
F: 

$\underset{\sim}{x} \in \mathbb{R}^n \longmapsto \underset{\sim}{y} \in \mathbb{R}^n$



$F(\underset{\sim}{x}_1) = F(\underset{\sim}{x}_2) \Rightarrow \underset{\sim}{x}_1 = \underset{\sim}{x}_2$

onto "every $\underset{\sim}{y} \in F(D)$ is hit"



$F(D) = \{ \underset{\sim}{y} : \underset{\sim}{y} = F(\underset{\sim}{x}) \text{ some } \underset{\sim}{x} \in D \}$

guaranteed F onto !

then F^{-1} exists: $\underset{\sim}{y} = F(\underset{\sim}{x})$ iff $\underset{\sim}{x} = F^{-1}(\underset{\sim}{y})$

- To construct the inverse of $w = e^z$, we find D such that

$$D \xrightarrow{\text{1-1 onto}} \mathbb{C}$$
$$z \longmapsto w$$

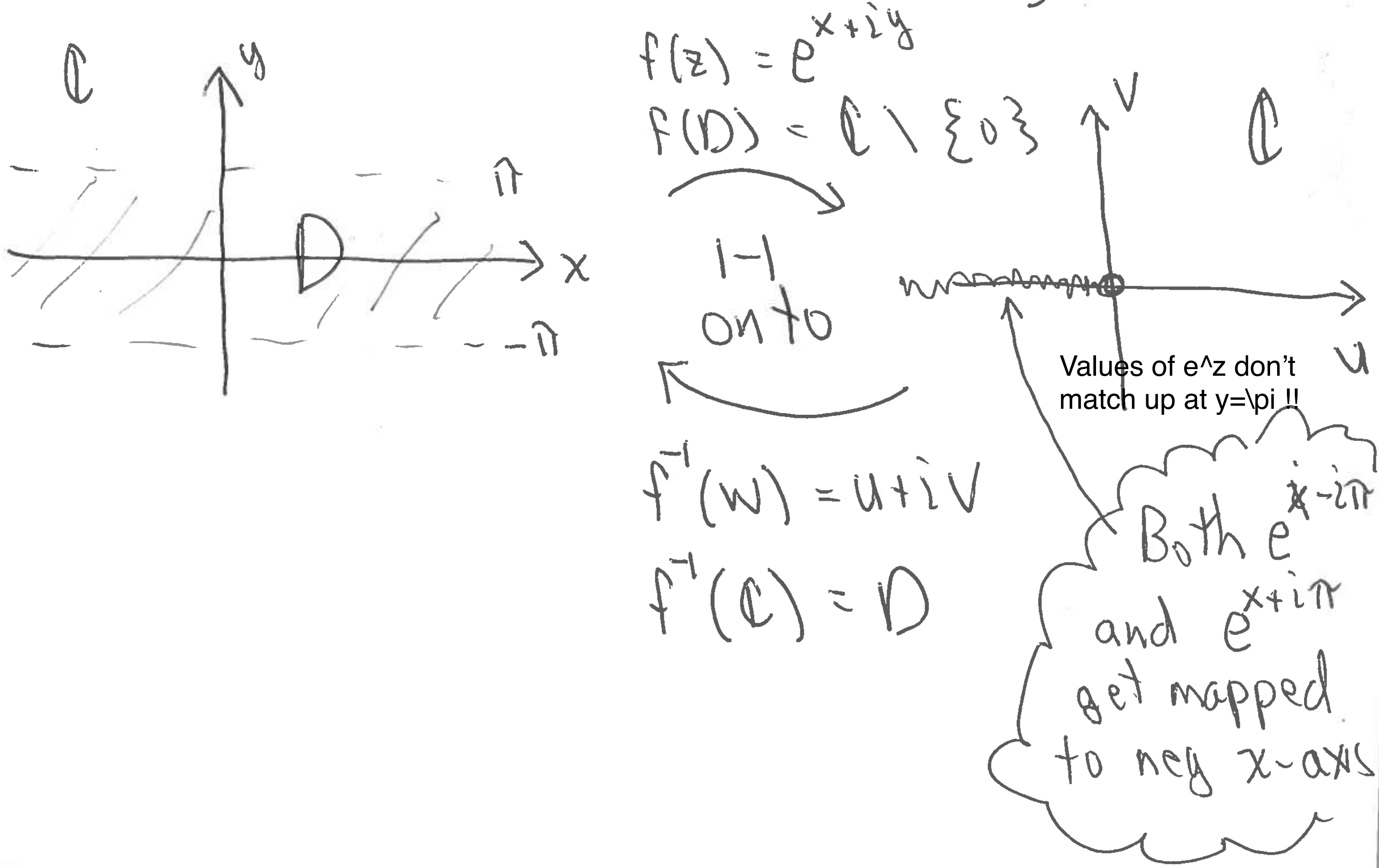
- We construct the inverse of $w = f(z) = e^z$

$$u + iv = w = f(z) = e^{x+iy}$$

We must restrict z to a domain $D \subseteq \mathbb{C}$ on which f is 1-1 onto $\mathbb{C} \setminus \{0\}$

Since e^z is 2π-periodic in y, restrict to

$$D = \{ z \in \mathbb{C} : x \in \mathbb{R},\ -\pi < y \le \pi \}$$



[4] To construct the inverse of

$$f: U \to \tilde{V} \quad \text{1-1 onto}$$

Start: $w = f(z)$

Switch z, w $\quad z = f(w)$

Solve for w; $\quad w = f^{-1}(z) \quad f^{-1}: \tilde{V} \to U$

- So start with $u + iv = e^x e^{iy} \Leftrightarrow w = e^z$

Switch w & z: $\quad x + iy = e^u e^{iv} \Leftrightarrow z = e^w$

Solve for w: $\quad |z| = e^u \quad u = \ln|z|$

$$z = re^{i\theta} \Rightarrow v = \mathrm{Arg}\{z\} \in (-\pi, \pi]$$

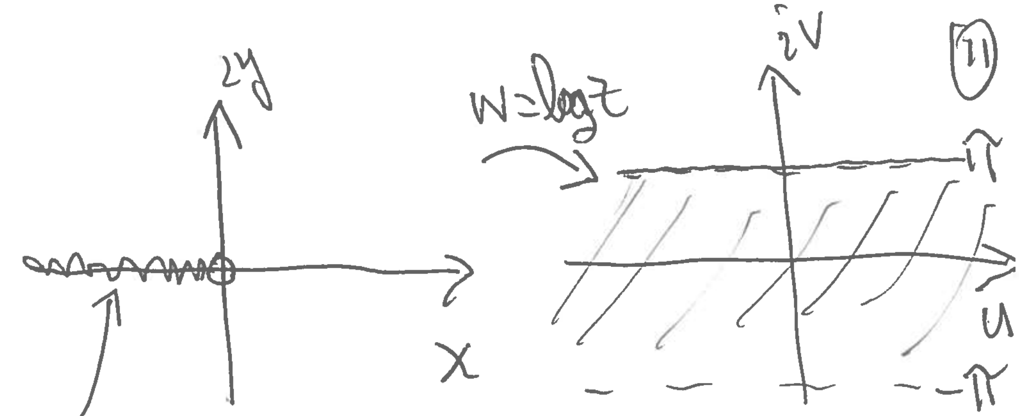
Conclude: $u + iv = \ln|z| + i\theta$

OR: $\boxed{z = re^{i\theta} \Leftrightarrow w = \ln r + i\theta} \boxed{= \log z}$

Define: $\log z = \ln r + i\theta$

$$\{z \in \mathbb{C} : z \neq 0\} \longrightarrow \{w \in \mathbb{C} : -\pi < v \leq \pi\}$$

log: $z \longmapsto w$



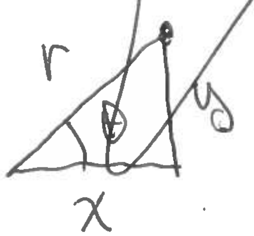
undefined at $z=0$, discontinuous along neg real axis

- Check: $w = \log z$ satisfies C-R eqns —

$$\text{Log}\, z = \underbrace{\ln r}_{u} + i\underbrace{\theta}_{v} \qquad (z = re^{i\theta})$$

$$u_x = \frac{\partial}{\partial x}\ln(r) = \frac{1}{r}\cdot\frac{x}{r} = \frac{x}{r^2}\ ; \quad u_y = \frac{y}{r^2}$$

$$\cancel{v_y = \frac{\partial \theta}{\partial y} = \frac{\partial}{\partial y}\text{Arcsin}\left(\frac{y}{r}\right) = \frac{1}{\sqrt{1-\left(\frac{y^2}{r^2}\right)}}\left(+ \ldots\right)}$$

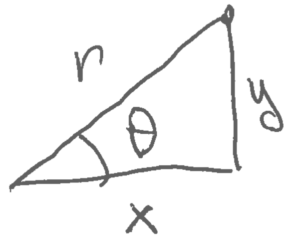


$$\cancel{\frac{y}{r} = \sin\theta}$$

$$\cancel{\theta = \text{Arcsin}\,\frac{y}{r}}$$

(wlog assume x>0. For other values use Arccos(x/r), etc.)

$$V_y = \frac{\partial \theta}{\partial y} = \frac{\partial}{\partial y}\,\text{Arcsin}\left(\frac{y}{r}\right) = \frac{1}{\sqrt{1-\left(\frac{y}{r}\right)^2}}\,\frac{\partial}{\partial y}\left(\frac{y}{r}\right)$$



$$= \frac{1}{\sqrt{1-\left(\frac{y}{r}\right)^2}}\left(\frac{1}{r} - \frac{y^2}{r^3}\right)$$

$\sin\theta = \frac{y}{r}$

$\theta = \text{Arcsin}\left(\frac{y}{r}\right)$

$$= \frac{\cancel{r}}{\sqrt{r^2-y^2}}\,\frac{1}{\cancel{r}}\left(1 - \frac{y^2}{r^2}\right)$$

$$= \sqrt{r^2-y^2}\cdot\frac{1}{r^2} = \frac{x}{r^2}$$

$$V_x = \frac{\partial \theta}{\partial x} = \frac{\partial}{\partial x}\,\text{Arcsin}\left(\frac{y}{r}\right) = \frac{1}{\sqrt{1-(y/r)^2}}\,\frac{\partial}{\partial x}\left(\frac{y}{r}\right)$$

$$= \frac{\cancel{r}}{\underset{x}{\cancel{\sqrt{r^2-y^2}}}}\cdot\left(-\frac{y}{r^2}\cdot\frac{\cancel{x}}{\cancel{r}}\right) = -\frac{y}{r^2}$$

__So__: $u_x = \frac{x}{r^2} = V_y$ ✓ $\quad u_y = +\frac{y}{r^2} = -V_x$ ✓

C-R holds $\Rightarrow \ln z = \ln r + i\theta$ is analytic on its domain $\mathbb{C} \setminus \{\text{negative Real axis } \theta = \pm\pi\}$

- Q: $\log z$ is differentiable, but what is its derivative?

Easiest to do this in general:

Assume $w = f^{-1}(z)$ and $z = f(w)$ are both differentiable...

Then: $$z = f(f^{-1}(z)) \quad \text{holds } \forall z \in \mathbb{C}$$

Diff both sides (both are the same fn of z)

$$1 = f'(\underbrace{f^{-1}(z)}_{w}) \cdot \frac{d}{dz} f^{-1}(z)$$

$$\Rightarrow \boxed{\frac{d}{dz} f^{-1}(z) = \frac{1}{f'(w)}}$$

"The derivative of the inverse f^{-1} at z is one over the deriv of f at w."

So assume $f^{-1}(z) = \log z$, $f(w) = e^w = z$

$f'(w) = f(w) = e^w = z$

Then $\dfrac{d}{dz} f^{-1}(z) = \dfrac{1}{f'(w)}$.

$$\frac{d}{dz} \log z = \frac{1}{\frac{d}{dw} e^w} = \frac{1}{e^w} = \frac{1}{z}$$

$$\boxed{\frac{d}{dz} \log z = \frac{1}{z}}$$

(off neg real axis where log is discont)

☐ Actually – we use standard properties of derivatives carry over to complex valued functions —

Thm: Assume f,g are differentiable at z.
Then:

(1) $\frac{d}{dz}(f+g) = f' + g'$ at z

(2) $\frac{d}{dz}\frac{f}{g} = \frac{gf' - fg'}{g^2}$ at z, $g(z) \neq 0$

(3) $\frac{d}{dz}(K \cdot f) = K f'$

(4) Chain Rule: $\frac{d}{dz} f(g(z)) = f'(g(z))g'(z)$
(so long as $g'(z)$ exists & $f'(w)$ exists where $w = g(z)$)

<u>Proof</u>: Since $\mathbb{C}$ is a field, exactly the same argument carries over from the real case.

Eg: Chain Rule

$$\frac{d}{dz}(f\circ g)(z) = \lim_{\Delta z \to 0} \frac{f(g(z+\Delta z)) - f(g(z))}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{f(g(z+\Delta z)) - f(g(z))}{g(z+\Delta z) - g(z)} \cdot \frac{g(z+\Delta z) - g(z)}{\Delta t}$$

Let $g(z) = w$, $g(z+\Delta z) = w + \Delta w$ so $\Delta z \to 0$
$\Rightarrow \Delta w \to 0$

$$= \lim_{\Delta z \to 0} \frac{f(w+\Delta w) - f(w)}{\Delta w} \cdot \frac{g(z+\Delta z) - g(z)}{\Delta t}$$

$= f'(w)\, g'(z)$ "limit of the product is the product of the limits"

Note: Since $z \to z_0 \iff (x,y) \to (x_0, y_0)$,
all the properties of limits in $\mathbb{R}^2$
carry over to limits in $\mathbb{C}$

Note: g diff at $z \Rightarrow g$ cont at z

so $\Delta z \to 0 \Rightarrow \Delta w \to 0$ ✓

Note: We still have to worry about
possibity that $g(z+\Delta z) - g(z) = 0$!
If this happens in a nbhd of z, take
$z_n \to z$ with $g(z_n) - g(z) = 0$
$\Delta z = z_n - z \Rightarrow \frac{d}{dz}(f \circ g)(z) = 0$ since
limit is same every way $\Delta z \to 0$!
OW restrict to nbhd of z in which $g(z+\Delta z) - g(z) \neq 0$