IV

Complex Exponents

[3] The function $f(z) = z^a$ for $a \in \mathbb{R}$.

[2] Consider first DeMoivres Formula

$n \in \mathbb{N}$ $\quad e^{in\theta} = \cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n = (e^{i\theta})^n$

Proof: That $e^{i(\theta_1+\theta_2)} = e^{i\theta_1}e^{i\theta_2}$ follows by the sum angle formulas for sine & cosine. Ie

$$e^{i(\theta_1+\theta_2)} = \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)$$

$$= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\cos\theta_1\sin\theta_2 + \cos\theta_2\sin\theta_1)$$

$$= (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = e^{i\theta_1}e^{i\theta_2}.$$

Therefore: $e^{i2\theta} = e^{i\theta}e^{i\theta} = (e^{i\theta})^2$

Assume $n-1$: $e^{in\theta} = e^{i(n-1)\theta + i\theta} = e^{i(n-1)\theta}e^{i\theta}$
(for induction)

$$= (e^{i\theta})^{n-1}e^{i\theta} = (e^{i\theta})^n$$

$(n-1) \Rightarrow (n)$ so pf by induction is complete.

By DeMoivres Formula,

$$z^n = \underbrace{(re^{i\theta})^n}_{\text{product } n \text{ times}} = r^n (e^{i\theta})^n = r^n e^{in\theta}$$

$$\boxed{z^n = r^n e^{in\theta}}$$

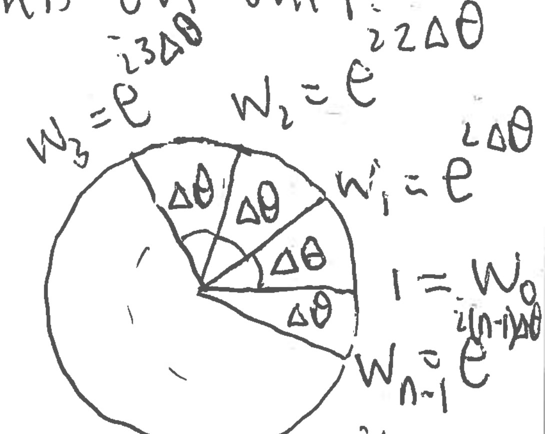
(b) Assume $a = \frac{1}{n}$, $n \in \mathbb{N}$. $z^{1/n}$ are the n'th roots of z.

Defn: We say $w = z^{1/n}$ if $w^n = z$

- Case 1: $z = 1$. Then $w^n = 1 \Rightarrow w_k = e^{\frac{i2k\pi}{n}}$ $\quad k = 0, 1, \dots, n-1$

Ie., $w_k^n = \left(e^{\frac{i2k\pi}{n}}\right)^n = e^{i2k\pi} = 1$ ✓

$w_k = e^{\frac{i2k\pi}{n}}$, $k = 0, \dots, n-1$ are the n points on unit circle equidistant around...



Ie., $\Delta\theta = \frac{2\pi}{n}$, $\theta_k = k\Delta\theta$

$k = 0, \dots, n-1$

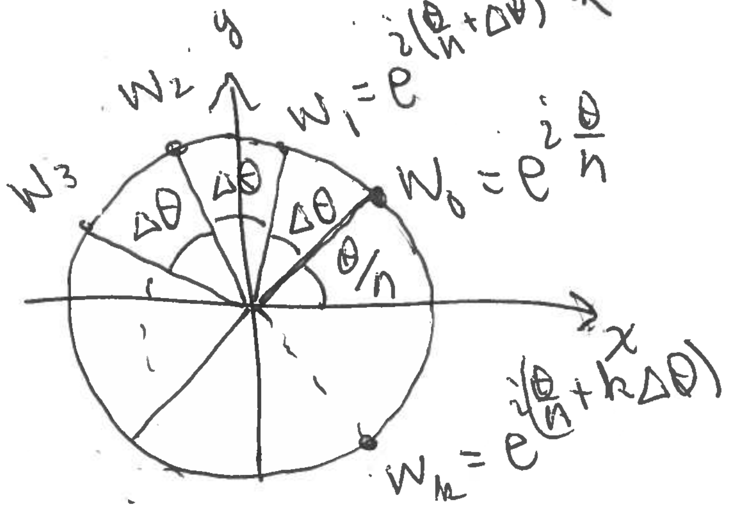
$w_0 = 1, w_1 = e^{i\Delta\theta}, \dots w_{n-1} = e^{i(n-1)\Delta\theta}$ are the n roots of unity

• Case 2: $z = e^{i\theta}$ Then $w^n = e^{i\theta} \iff w_k = e^{i\frac{\theta + 2k\pi}{n}}$

$k = 0, \dots, n-1$

So $w_k = \underbrace{e^{i\frac{\theta}{n}}}_{\text{angle theta betw zero \& } 2\pi} \; \underbrace{e^{i\frac{2k\pi}{n}}}_{\text{These are the n roots of unity}}$

Let $\Delta\theta = \frac{2\pi}{n}$, $\theta_k = k\Delta\theta$, $w_k = e^{i\left(\frac{\theta}{n} + k\Delta\theta\right)}$



n-points on unit circle equidistant starting at θ/n

Then $w_k = e^{i(\theta_0 + k\Delta\theta)}$ $k = 0, \dots, n-1$ are the n-roots of $e^{i\theta}$

• Case 3: $z = re^{i\theta}$ $\quad w^n = re^{i\theta} \iff w_k = r^{1/n} e^{i\frac{\theta + 2k\pi}{n}}$

$w_k = r^{1/n} \underbrace{e^{i(\theta_0 + k\Delta\theta)}}_{n\text{-roots of } e^{i\theta_0}}$

new length takes it off unit circle.

iv Assume $a = m/n \in \mathbb{Q}$ rationals, $m, n \in \mathbb{Z}$

If $\frac{m}{n}$ negative, define $z^a = \frac{1}{z^{|a|}}$

So assume $m, n > 0$.

Define: $z^{m/n} = \left(z^{1/n}\right)^m$

$z^{1/n} = w_0, w_1, \dots, w_{n-1}$ $\quad w_k = r^{1/n} e^{i\left(\frac{\theta + k2\pi}{n}\right)}$

Then n roots of $e^{i\theta}$

$\Rightarrow z^{m/n} = w_0^m, w_1^m, \dots, w_{n-1}^m$ $\quad w_k^m = r^{m/n}\left(e^{i\frac{\theta + k2\pi}{n}}\right)^m$

Conclude: $w = z^{m/n}$ has n solutions if $\left[\frac{m}{n}\right] = 1$ lowest terms

v Finally – what if a is irrational? What is z^{π} for example? It should have many solutions, but how many? We could try to make sense of $q_i \longrightarrow \pi$ and define $z^{\pi} = \lim_{i \to \infty} z^{q_i}$ $\quad q_i \in \mathbb{Q}$. Cleaner way:

(iii) Define z^a for a irrational

<u>Defn</u>: $z^a = (e^{\log z})^a \equiv e^{a \log z}$

$$\boxed{z^a = e^{a \log z}}$$

Now for each branch of the logarithm, we get one solution. As we take

$$2\pi(k-1) \leq \theta \leq 2\pi k \qquad k = 1, \ldots, \infty$$

we get all the possible solutions ~~of~~ $w = z^a$.

For $a = \frac{m}{n}$ in lowest terms, the solutions <u>repeat</u> at $k = n$, so we get n distinct soln's. For a irrational, we get ∞ # of different solutions.

Example: If $a = m/n \in \mathbb{Q}^+$, show that n solutions $z^a = z^{m/n}$ defined above, each correspondi to a particular branch of the logarithm in $z^a = e^{a \log z}$

Soln: $z^{m/n} \underset{\text{Def}}{\equiv} e^{\frac{m}{n}\log z} = e^{\frac{m}{n}(\ln r + i(\theta + 2k\pi))}$

choice of k is a choice of branch for logarithm (ie a choice of name for θ)

$= e^{\frac{m}{n}\ln r} e^{im\frac{(\theta+2k\pi)}{n}}$

$= r^{m/n} \underbrace{\left(e^{i\frac{\theta+2k\pi}{n}}\right)^m}$

these are the n roots of $e^{i\theta}$, different for $k = 0, \dots, n-1$

$= w_k^m$ where $w_0^m, \dots, w_{n-1}^m$ are the n soln's $z^{m/n}$, where w_k are the n roots of z. ✓

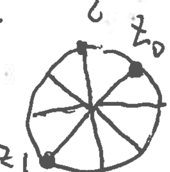
Finally, for $a \in \mathbb{C}$, we can define

$$z^a = e^{a \log z} \qquad (*)$$

Each choice of branch for $\log z$ determines a unique value for z^a.

$\underline{\underline{Ex}}$: (HW) Show when $a = 1/2$ & $z = i$, $(*)$ reproduces the two square roots of i

$\underline{\underline{Soln}}$: $w = (i)^{1/2}$ $\Rightarrow z_0 = e^{i\frac{\pi}{4}}, \; z_1 = e^{i\frac{5\pi}{4}}$



$\Leftrightarrow \quad w^2 = i = e^{i\pi/2}$

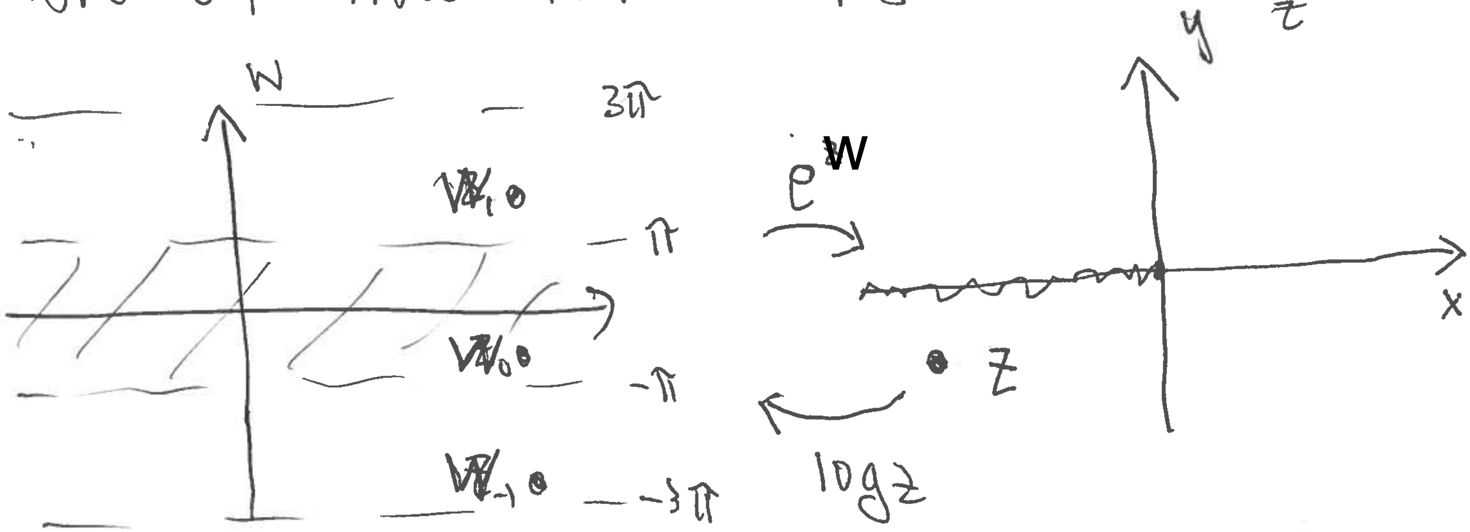
$\Leftrightarrow \quad w_0 = e^{i\pi/4}, \; w_1 = e^{i\frac{\pi/2 + 2\pi}{2}} = e^{i\frac{5\pi}{4}} \qquad \left(\frac{5\pi}{4} = \frac{\pi}{4} + \pi\right)$

Check: $z^{1/2} = e^{\frac{1}{2}\log z} = e^{\frac{1}{2}\log(i)} = e^{\frac{1}{2}(\ln 1 + i(\pi/2 + 2k\pi))}$ $\quad k = 0, 1$

$= e^{\frac{1}{2} i(\pi/2 + 2k\pi)} = e^{i\left(\frac{\pi}{4} + k\pi\right)} = \underset{k=0}{e^{i\pi/4}}, \; \underset{k=1}{e^{i\frac{5\pi}{4}}}$

Conclude: $z^{1/2} = e^{\frac{1}{2}\log z}$ defines each branch of the sq root in terms of branches of $\log z$ ✓

Note on how this works:



$\log z$ is the inverse of e^z

$$w = \log z \iff z = e^w$$

$$z = e^{\log z} = e^{\ln|z| + 2\pi k i}$$

this holds for any choice of branch

$$w = \log e^w$$

This requires branch of log contain w

$z = e^{\log z}$ (always!)

~~if we choose branch of log z to include z~~. If we put in all the the pts that e^w maps to z we get

$$\log z = \ln|z| + 2\pi k i = \{w_k\}_{k=-\infty}^{\infty} \qquad e^{w_k} = z \ \checkmark$$

→ When we write

$$z^a = e^{a \log z} = e^{a(\ln|z| + \underbrace{\cancel{2\pi k} i}_{\text{we get all the pre-images of } z})}$$

$\theta_0 + 2\pi k$

Multiplication by a then changes all the $w_k \Rightarrow$ different # for every k, or else a finite # if $a \cdot k$ is commensurate with 2π (the case $a = \frac{m}{n}$)

- $f(z) = z^a$ as an analytic function —

By the formula $z^a = e^{a \log z} = f(z)$, f is the composition of analytic fn's $f(z) = g \circ h(z)$ on the branch where $\log z$ is defined & analytic (i.e., in $\mathbb{C} \setminus \{$Neg Real axis, for $\mathrm{Arg}(z) \in (-\pi, \pi]\}$)

$g(u) = e^u$, $h(z) = a \log z$
$u = a \log z$

Thus: $\frac{df}{dz} = \frac{dg}{du} \cdot \frac{dh}{dz} = e^u \cdot a \cdot \frac{1}{z} = z^a \cdot a \cdot \frac{1}{z}$

$$\Rightarrow \boxed{\frac{d}{dz}(z^a) = a z^{a-1}}$$ as in the real case ✓

The inverse function Theorem

Recall a complex function

$$w = f(z) \quad f: \mathbb{C} \to \mathbb{C}$$

$$w = u(x,y) + i v(x,y)$$

f corresponds to the vector valued function

$$\vec{f} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$(x,y) \longrightarrow (u(x,y), v(x,y)) = \vec{f}(x,y)$$

Recall the Jacobian of $\vec{f}$:

$$J = D\vec{f} = \begin{bmatrix} -\nabla u- \\ -\nabla v- \end{bmatrix} = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$$

Assume f in C^1:

By C-R we know f diff iff $u_x = v_y, u_y = -v_x$

$$\Leftrightarrow \quad D\vec{f} = \begin{bmatrix} u_x & u_y \\ -u_y & u_x \end{bmatrix} \equiv J(x_0, y_0)$$

<u>The Inverse Function Theorem</u>: We recall ~~as~~ when a function $\vec{f} : \mathbb{R}^2 \to \mathbb{R}^2$ $\overrightarrow{(x,y)} \mapsto \overrightarrow{(u,v)}$ has an inverse in a nbhd (neighborood $\equiv$ open ball) centered at $\overrightarrow{(x_0,y_0)}$. The real (IFT) is proven in Mat 127B. We assume it, and use it to prove the Complex inverse fn thm for $f: \mathbb{C} \to \mathbb{C}$ $x+iy \mapsto u(x,y)+iv(x,y)$. The IFT requires u, v satisfy the minimum regularity i.e. $\vec{f}$ must have continuous derivative in a nbhd of $\underset{\sim}{x}_0 = \overrightarrow{(x_0,y_0)}$. ($u_x, u_y, v_x, v_y$ must be cont fn's in a nbhd of $\underset{\sim}{x}_0$. We say $\vec{f} \in C^1$)

<u>In fact</u>, analytic f will always have an ∞ # of continuous derivatives, so we assume at the start that $f: \mathbb{C} \to \mathbb{C}$ is analytic and $u, v \in C^1$ in some nbhd of $\underset{\sim}{x}_0$.

First we state the real version of IFT which we assume here

Inverse Fn Thm (real case Mat 127B) Assume $\vec{f}:\mathbb{R}^2\to\mathbb{R}^2$, $\vec{f}(x,y)=\overrightarrow{(u(x,y),v(x,y))}$ and assume $\vec{f}\in C^1$ in a nbhd of $\underset{\sim}{x}_0=\overrightarrow{(x_0,y_0)}$. Assume further that

$$(*)\quad \text{Det } J(x_0,y_0)\equiv\begin{vmatrix}-\nabla u_0-\\-\nabla v_0-\end{vmatrix}=\begin{vmatrix}u_x & u_y\\ v_x & v_y\end{vmatrix}_{\underset{\sim}{x}_0}\equiv u_xv_y-u_yv_x\Big|_{\underset{\sim}{x}_0}\neq 0.$$

Then $\vec{f}$ has an inverse $\vec{f}^{-1}$ in a nbhd of $\underset{\sim}{x}_0$. I.e. $\exists\ \vec{f}^{-1}$ st

$$f^{-1}\circ f(\underset{\sim}{x})=\underset{\sim}{x}\ \ \&\ \ f\circ f^{-1}(\underset{\sim}{u})=\underset{\sim}{u}.$$

Proof (Mat 127B)

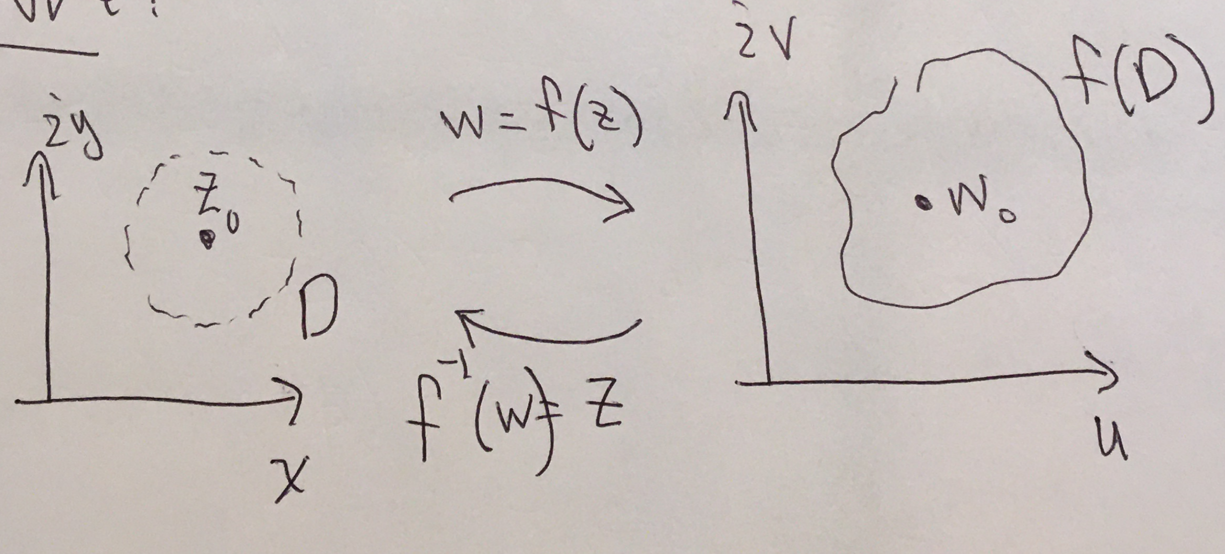
Conclude: if $f(z)$ is analytic at $\underset{\sim}{x}_0\equiv x_0+iy_0$, then $\vec{f}(\underset{\sim}{x}_0)=\underset{\sim}{u}_0$, $\text{Det } J(x_0,y_0)=\begin{vmatrix}u_x & u_y\\ -u_y & u_x\end{vmatrix}_{\underset{\sim}{x}_0}=u_x^2+u_y^2\Big|_{\underset{\sim}{x}_0}\neq 0$ iff $f'(z_0)\neq 0$. Thus the condition $f'(z_0)\neq 0$ guarantees condition $(*)$ of the real IFT $\blacksquare$

Thm: The complex Implicit Function Thm:
Assume f is analytic & $u, v \in C^1$ in a nbhd of $z_0 = x_0 + iy_0$ and assume $f'(z_0) \neq 0$. Then f has an inverse f^{-1} in a nbhd of z_0, with

$$f^{-1} \circ f(z) = z, \quad f \circ f^{-1}(w) = w$$

$$z \in D \qquad\qquad w \in f(D)$$

Picture:



Q: Is the inverse always diff at w_0?
I.e., does f^{-1} satisfy the C-R equn's @ z_0?

Thm: If f is a cont diff at z_0 & $f'(z_0) \neq 0$, then f^{-1} exists in a nbhd of z_0 & f^{-1} is diff @ w_0 & $f^{-1\prime}(w_0) = \frac{1}{f'(z_0)}$

Proof: We have

$$w = f(z) \text{ iff } z = f^{-1}(w)$$

$$(u,v) = \vec{f}(x,y) \qquad (x,y) = \vec{f}^{-1}(u,v)$$

$$\underset{\sim}{w} = \vec{f}(\underset{\sim}{x}) \text{ iff } \underset{\sim}{x} = \vec{f}^{-1}(\underset{\sim}{w})$$

Thus

$$\vec{f}^{-1}(\vec{f}(\underset{\sim}{x})) = \underset{\sim}{x}$$

Differentiate by chain Rule:

$$\underset{2\times 2}{\frac{\partial \vec{f}^{-1}}{\partial \underset{\sim}{w}}} \circ \underset{2\times 2}{\frac{\partial \underset{\sim}{w}}{\partial \underset{\sim}{x}}} = \underset{2\times 2}{I} \equiv \text{identity matrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$D\vec{f}^{-1}\, Df = I$$

Thus: $Df^{-1} = J^{-1} = \begin{bmatrix} u_x & u_y \\ -u_y & u_x \end{bmatrix}^{-1}$

$$= \frac{1}{\mathrm{Det}[J]} \cdot \begin{bmatrix} u_x & \boxed{-u_y} \\ \boxed{u_y} & u_x \end{bmatrix}.$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$ check!

$$J_{f^{-1}} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \qquad \begin{array}{l} x_u = y_v = \frac{u_x}{\mathrm{Det}\,J} \\ \quad\quad C\text{-}R \\ x_v = -y_u = -\frac{u_y}{\mathrm{Det}\,J} \end{array}$$

f^{-1}: $x(u,v) + i\,y(u,v) = f^{-1}(u,v)$

CR $\boxed{\begin{array}{l} x_u = y_v \\ x_v = -y_u \end{array}} \Rightarrow f^{-1}$ Diff.

- Since $f^{-1}(f(z)) = z \underset{\text{chn Rule}}{\Rightarrow} f^{-1\,\prime}(w) \cdot f'(z) = 1$

$$\Rightarrow f^{-1\,\prime}(w_0) = \frac{1}{f'(z_0)}$$

HW-FIP

☒ Using FTC to prove

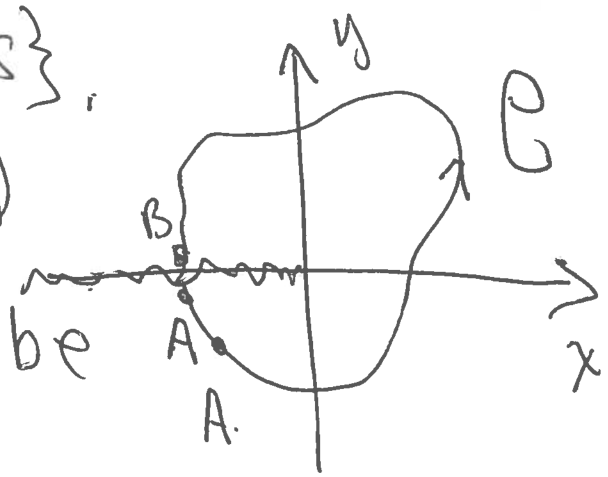
$$\oint_{\mathcal{C}} \frac{dz}{z} = 2\pi i$$

$\mathcal{C}$ any closed curve surrounding $z=0$ counterclockwise

Pf. We have: If $f(z)$ is differentiable in a simply connected region $D \subseteq \mathbb{C}$, then $F(z)$ exists such that $F'(z) = f(z)$, all anti-derivatives of f agree with F to within a constant, and $\int_A^B f(z)\,dz = F(B) - F(A)$.

We have: $F(z) = \log z$ satisfies $F'(z) = \frac{1}{z}$ in $\mathbb{C} \setminus \{\text{negative real axis}\}$.

This $\underline{\underline{\text{is}}}$ a simply connected region (closed curves can be contracted to a point.)



So let $A = B$ be the point in $\mathbb{C}$ where curve $\mathcal{C}$ crosses the negative real axis.

Then

$$\left.\begin{aligned} A = A^- &= re^{-i\pi} \\ B = A^+ &= re^{+i\pi} \end{aligned}\right\}$$

same point, different names, but $\log z$ reads
$\log A^- = \ln r - i\pi$
$\log A^+ = \ln r + i\pi$

Thus —

$$\oint_{\mathcal{C}} \frac{dz}{z} = \int_{A^-}^{A^+} \frac{dz}{z} = F(A^+) - F(A^-) \qquad F(z) = \log z$$

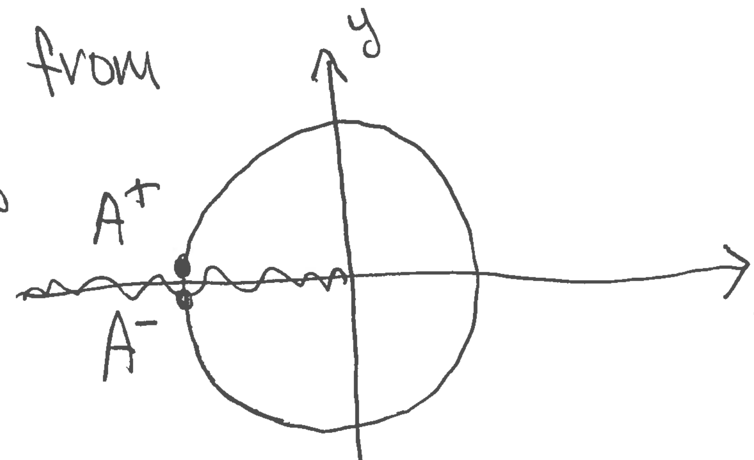
$$= \log A^+ - \log A^-$$

$$= \cancel{\ln r} + i\pi - \cancel{\ln r} - (-i\pi)$$

$$= 2\pi i \checkmark$$

HW: Use the theory here to define the complex inverse sine Arcsin(w), and find a formula for its derivative.

Note: $\log z$ has limits from above & below on the branch cut, but is discontinuous there!



HW: Define the complex inverse sine Arcsin(w) and determine its derivative.