IX

Calculating Residues —

Recall: f analytic in nbhd of z_0 $\Rightarrow$

Taylor Series: $f(z) = \sum_{k=0}^{\infty} c_k (z-z_0)^k$

$$c_k = \frac{f^{(k)}(z_0)}{k!}$$

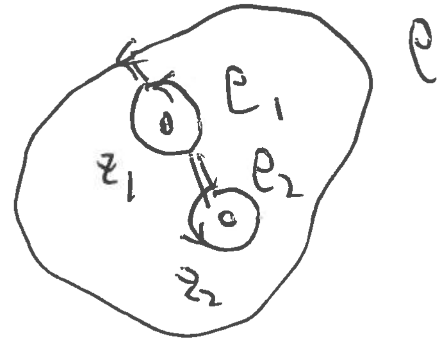
Converges in $0 \le |z-z_0| < R$, uniformly in $|z-z_0| < r < R$ where R is largest radius with no singularity of f in $|z-z_0| < R \Leftrightarrow R =$ dist to nearest singularity

Laurent Series:

$$f(z) = \sum_{k=1}^{\infty} c_{-k} (z-z_0)^{-k} + \sum_{k=0}^{\infty} c_k (z-z_0)^k$$

Converges in the largest annulus $r < |z-z_0| < R$ in which f is analytic, uniformly on every compact subset.

• Residue Thm –



$$\int_C f(z)\,dz = \sum_i \int_{C_i} f(z)\,dz$$

$$= 2\pi i \sum_{i=1}^{n} R(f; z_i)$$

applies when f has a finite # of pt singularities. Here

$$R(f; z_i) = C_{-1}$$

in the Laurent expansion of f at z_i.

Q: How to compute C_{-1} at z_i?

~~In fact, a non constant analytic fn contains only isolated zeros~~

Recall – a meromorphic function has only finite # of isolated pole singularities: Pole of order n at z_0 if $C_{-n} \neq 0$, $C_{-n-k} = 0$ $k \geq 1$.

- Simple Poles – Assume z_0 a simple pole:

$$f(z) = \frac{c_{-1}}{z - z_0} + \sum_{k=0}^{\infty} c_n (z - z_0)^k \qquad c_{-1} \neq 0$$

Conclude:

Thm ①: $c_{-1} = \lim_{z \to z_0} f(z)(z - z_0)$ for a simple pole

To check that f has a simple pole at $z = z_0$ all you need to check is:

$$\lim_{z \to z_0} f(z)(z - z_0) \neq 0$$

$$\lim_{z \to z_0} f(z)(z - z_0)^k = 0 \quad k \geq 1.$$

Simple poles are easy.

Eg: $f(z) = \dfrac{1}{(z-i)(z+i)^2}$ has simple pole @ $z = i$.

- Pole of order n @ $z = z_0$

$$f(z) = \frac{C_{-n}}{(z-z_0)^n} + \frac{C_{-n+1}}{(z-z_0)^{n-1}} + \cdots + \frac{C_{-1}}{z-z_0} + \sum_{k=0}^{\infty} C_k (z-z_0)^k$$

<u>Thm②</u> f has a pole of order n @ $z=z_0$ iff

$$\lim_{z\to z_0} f(z)(z-z_0)^n = C_{-n} \neq 0$$

$$\lim_{z\to z_0} f(z)(z-z_0)^{n+k} = 0 \qquad k \geq 1$$

Q: how to find C_{-1}?

Consider:

$$g(z) = f(z)(z-z_0)^n = C_{-n} + C_{-n+1}(z-z_0) + \cdots + C_{-1}(z-z_0)^{n-1} + \sum_{k=0}^{\infty} C_k (z-z_0)^{k+n}$$

$$\frac{d^{(n-1)}}{dz^{n-1}} g(z) = (n-1)!\, C_{-1}$$

get this by diff $n-1$ times

$$\boxed{C_{-1} = \frac{g^{(n-1)}(z_0)}{(n-1)!} = R(f; z_0)}$$

Not so easy to use when $n \geq 2$

<u>Example</u>(1) Find $I = \int_0^\infty \frac{x^2\,dx}{(x^2+9)(x^2+4)^2}$

<u>Soln</u>: $I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+9)(x^2+4)^2}$.

Let $f(z) = \frac{z^2}{(z^2+9)(z^2+4)^2} = \frac{z^2}{(z-3i)(z+3i)(z-2i)^2(z+2i)^2}$

Simple poles: $z = \pm 3i$

Double poles: $z = \pm 2i$

$R(f;3i) = \lim_{z\to 3i} f(z)(z-3i) = \frac{(3i)^2}{(3i+3i)(3i-2i)^2(3i+2i)^2}$

$= -\frac{3}{50i}$

To find residue at $z = 2i$ write

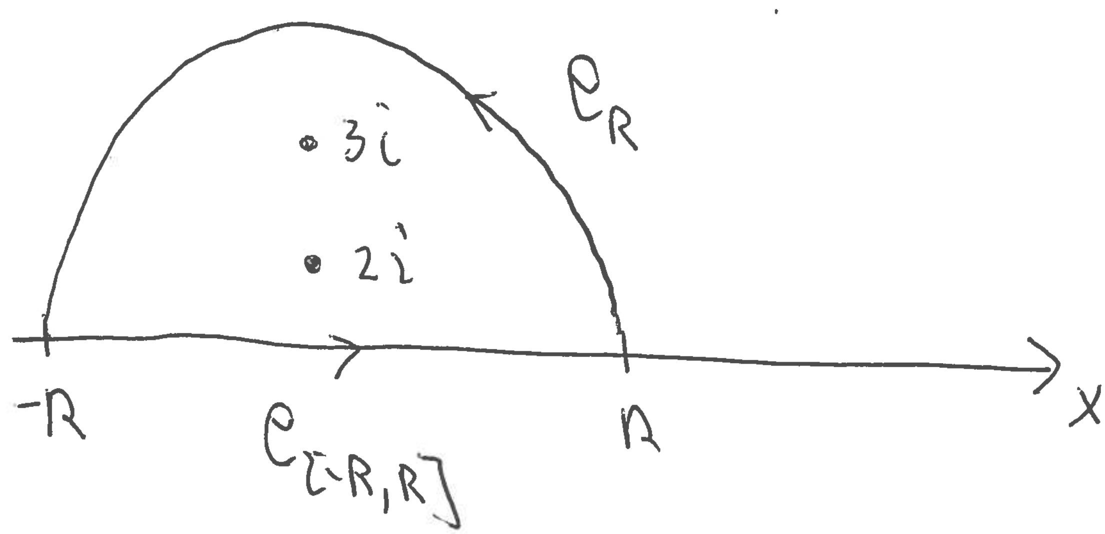
$$g(z) = (z-2i)^2 f(z) = \frac{z^2}{(z^2+9)(z+2i)^2}$$

$$R(f;2i) = g'(2i)$$

$$= \frac{(-4+9)(2i+2i)^2(4i) - (2i)^2\left[5(2)(4i)+(4i)^2(4i)\right]}{5^2(4i)^4}$$

$$\overset{\text{calc}}{=} -\frac{13i}{200}$$

Let $\mathcal{C}_R \equiv$ upper half of circle of radius R, & $\mathcal{C}_{[-R,R]} \equiv$ real line belw $-R \le x \le R$



Then Residue Thm $\Rightarrow$

$$\int_{C_{[-R,R]}+\mathcal{C}_R} f(z)\,dz = 2\pi i\left[R(f;3i) + R(f;2i)\right]$$

$$\int_{C_{[-R,R]}+\mathcal{C}_R} f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{\mathcal{C}_R} f(z)\,dz$$

want limit as $R\to\infty$ of this!

we show limit as $R\to\infty$ of this is zero

$$\left|\int_{\mathcal{C}_R} f(z)\,dz\right| \le \int_{\mathcal{C}_R} |f(z)|\,|dz| \le |\mathcal{C}_R| \max_{\mathcal{C}_R} |f(z)|$$

will test you on this!

will test you on this!

$$|f(z)| = \left| \frac{z^2}{(z^2+9)(z^2+4)^2} \right|$$

$$\leq \frac{|z|^2}{|z^2+9||z^2+4|^2}$$

$$\leq \frac{R^2}{(R^2-9)(R^2-4)^2} \qquad (R > 9)$$

$$= \frac{R^2}{R^6 (1 - 9/R^2)(1 - 4/R^2)^2}$$

$$\leq \frac{1}{4} \frac{1}{R^4} \qquad \left(\frac{9}{R^2} < \frac{1}{2}, \ \frac{4}{R^2} < \frac{1}{2} \right)$$

$$|C_R| = 2\pi R$$

$$\therefore \left| \int_{C_R} f(z)\,dz \right| \leq 2\pi R \cdot \frac{1}{4} \frac{1}{R^4} = \frac{1}{2}\pi \frac{1}{R^3} \xrightarrow[R \to \infty.]{} 0$$

Conclude:

$$\int_{\mathcal{C}} f(z)\,dz = 2\pi i\left(-\frac{3}{50i} - \frac{13i}{200}\right)$$

$$= \pi/100$$

$\uparrow$

calc

$$\therefore \lim_{R\to\infty} \int_{\mathcal{C}} f(z)\,dz = \frac{\pi}{100}$$

$$= \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx + \lim_{R\to\infty} \int_{\mathcal{C}_R} f(z)\,dz = \frac{\pi}{100}$$

$$\Rightarrow \boxed{\int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi}{100}}$$

Note: Many of the functions we apply residue thm to look like

$$f(z) = \frac{g(z)}{h(z)}$$

where g & h are analytic fns.

Thus singularities of f are the zeros of h. Thus its nice to know the following —

<u>Thm ①</u>: The zeroes of an analytic fn are always isolated

<u>Thm ②</u>: $f(z) = \frac{g(z)}{h(z)}$ always ~~has~~ is meromorphic, ie. sing's of f isolated at zeros of h are always <u>poles</u>.

– follows from T-series –

☐ Zeroes of an analytic fn —

Assume f analytic in a nbhd of z_0

Then T-series converges —

$$f(z) = \sum_{k=0}^{\infty} C_n (z - z_0)^k$$

Now $f(z_0) = 0 \Rightarrow C_0 = 0$. But $f(z) \neq 0$ implies some $C_n \neq 0$. Let C_n be the first non-zero C_n. Then

$$(z-z_0)^n f(z) = (z-z_0)^n \sum_{k=0}^{\infty} C_{k+n} (z-z_0)^n$$

$C_n \neq 0$

In this case we say f has a zero of order $n \Rightarrow \frac{1}{f(z)}$ has a pole of order n.

Conclude. f, g analytic & $g \neq 0$ on C. $\Rightarrow$

$$\int_C \frac{f(z)}{g(z)} dz = 2\pi i \sum R_i$$

R_i is residue @ z_i, z_i are the zeroes of g inside R.

Thus to prove $f(z) \neq 0$ in a nbhd of a zero of order n, it suffices to prove $\sum_{k=0}^{\infty} C_{k+n}(z-z_0)^k$ is non-zero in a deleted nbhd of z_0. This follows because

$$\sum_{k=0}^{\infty} C_{k+n}(z-z_0)^k = C_n + (z-z_0)\underbrace{\left\{\sum_{k=1}^{\infty} C_{k+n}(z-z_0)^{k-1}\right\}}$$

analytic $\Rightarrow$ cont $\Rightarrow$ bded in a nbhd of z_0 by M

$=0$ ⊗ $z \to z_0$ b/c cont

$$\left|\sum_{k=0}^{\infty} C_{k+n}(z-z_0)^k\right| \geq |C_n| - \left|\sum_{k=1}^{\infty} C_{k+n}(z-z_0)^k\right|$$

$$\left|\sum_{k=0}^{\infty} C_{k+n}(z-z_0)^k\right| \geq |C_n| - M|z-z_0| > 0$$

for $|z-z_0| < \dfrac{|C_n|}{M}$, $M = \text{Max}\,|\{\ \}|$

Theorem (Important) An analytic function is determined by its value on any convergent infinite sequence of distinct pts $\{z_n\}_{n=1}^{\infty}$. That is, if $f(z_n)$ is assigned, then this determines f. (Assume this without proof)

Corollary: An ~~nonzero~~ analytic function can only have a finite # of zeros inside a scc $\mathcal{C}$.

Proof: Let $\bar{\mathcal{C}}$ denote the closure of the inside of $\mathcal{C}$. $\bar{\mathcal{C}}$ is a compact set. Let $\{z_n\}$ be the zeroes of f inside $\bar{\mathcal{C}}$. If infinite $\Rightarrow$ $\exists$ convergent subsequence $z_n \to \bar{z} \in \bar{\mathcal{C}}$. But $f(z_n) = 0 \Rightarrow f(z) = 0$ $\forall$ z (since $f(z) \equiv 0$ is analytic & $\exists!$ analytic fn with these values).

Ex (2) Consider integrals of type

$$\int_0^{2\pi} F(\sin\theta, \cos\theta)\,d\theta. \qquad (1)$$

where $F(\sin\theta, \cos\theta) = \dfrac{p(\sin\theta, \cos\theta)}{q(\sin\theta, \cos\theta)}$

where $p(x,y)$, $q(x,y)$ are polynomials.

Then Let $z = e^{i\theta}$ and view θ as a parameter on the unit circle. In this case

$$\cancel{\sin\theta = \frac{z - \bar{z}}{2i}}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - z^{-1}}{2i}, \quad \cos\theta = \frac{z + z^{-1}}{2}$$

and $z(\theta) = e^{i\theta} \Rightarrow dz = ie^{i\theta}\,d\theta$.

~~Then contour integral (1) represents~~

Then (1) is the parameterization of a contour integral around unit circle

$$\int_0^{2\pi} F(\sin\theta, \cos\theta)\, d\theta = \oint_{C_0} F\left(\frac{z-z^{-1}}{2i}, \frac{z+z^{-1}}{2}\right)\underbrace{\left(-ie^{-i\theta}\right)}_{-i\frac{1}{z}} dz$$

Ex Evaluate

$$I = \int_0^{2\pi} \frac{d\theta}{\frac{5}{4} + \sin\theta}$$

$$\frac{5}{4} + \sin\theta = \underbrace{\frac{5}{4} + \frac{z - \frac{1}{z}}{2i}}_{\frac{5i + 2\frac{z^2-1}{z}}{4i}} = \frac{1}{4iz}\left(2z^2 + 5iz - 2\right)$$

$$= \frac{1}{4iz}\left(2z^2 + 5iz - 2\right) \checkmark$$

$$\therefore\ I = \int_{C_0} \frac{4\,dz}{2z^2 + 5iz - 2} = \int_{C_0} \underbrace{\frac{2\,dz}{(z+2i)(z+\frac{1}{2}i)}}_{f(z)}$$

15

$$I = 2\pi i \, R(f; -\tfrac{1}{2}i)$$

$$R(f; -\tfrac{1}{2}i) = \lim_{z \to -\frac{1}{2}i} f(z)\left(z + \tfrac{1}{2}i\right) = \frac{2}{-\frac{1}{2}i + 2i}$$

$$= \frac{4}{3i}$$

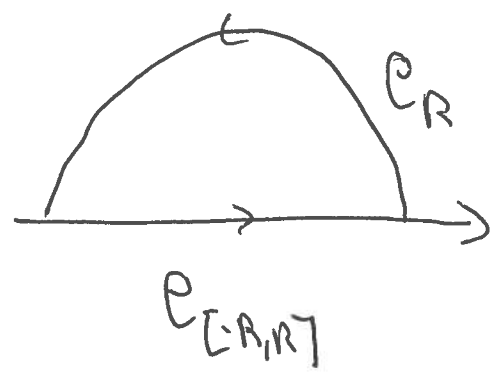
$$I = 2\pi \cancel{i} \cdot \frac{4}{3\cancel{i}} = \frac{8\pi}{3}$$ ✓

Ex (3) Improper Integrals involving sin/cos

$$\int_0^\infty \frac{\cos x \, dx}{x^2+1} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos x \, dx}{x^2+1}$$

Problem: $\displaystyle\int_{C} \frac{\cos z}{z^2+1}\,dz$ on $C_{[-R,R]} + C_R = C$

$$\cos z = \frac{e^{iz}+e^{-iz}}{2} = \frac{e^{-y}e^{ix}+e^{y}e^{-ix}}{2}$$



$e^{y} \to \infty$ on C_R as $R \to \infty$

Rather, set $\cos x = \mathrm{Re}\{e^{ix}\}$ so

$$I = \frac{1}{2}\,\mathrm{Re}\left\{\int_{-\infty}^{\infty} \frac{e^{ix}\,dx}{x^2+1}\right\}$$

upper $\frac{1}{2}$ plane bounded by 1

Consider now $\displaystyle f(z) = \frac{e^{iz}}{z^2+1} = \frac{e^{-y}e^{ix}}{z^2+1} \xrightarrow[\text{on } C_R]{0}$

$$\therefore \oint_{C} f(z)\,dz = 2\pi i \sum \underbrace{R(f;i)}_{e^{-1}\cdot\frac{1}{2i}} = \frac{\pi}{e} = \lim_{R\to\infty}\int_{C_{[-R,R]}} f(z)\,dz$$

Conclude:

$$\int_{\mathcal{C}_R + \mathcal{C}_{[-R,R]}} f(z)\,dz = \int_{\mathcal{C}_{[-R,R]}} f(z)\,dz + \underbrace{\int_{\mathcal{C}_R} f(z)\,dz}_{\xrightarrow{R\to\infty} 0}$$

$$= 2\pi i\, R(f; i) = 2\pi i \lim_{z\to i} f(z)(z-i)$$

$$= 2\pi i \frac{e^{i\cdot i}}{i+i} = \frac{\pi}{e}$$

$$\lim_{R\to\infty} \int_{\mathcal{C}_R + \mathcal{C}_{[-R,R]}} f(z)\,dz = \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1}\,dx + 0 = \frac{\pi}{e}$$

But $I = \frac{1}{2} \operatorname{Re}\left\{ \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1}\,dx \right\} = \boxed{\frac{\pi}{2e}}$