Summary:

$$\int_C f(z)\,dz = \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy$$

$$\vec{G}_1 = (u, -v) \qquad \vec{G}_2 = (v, u)$$

Assuming f satis C-R does 3 things –

① makes $f(z)$ differentiable

② makes $\text{Curl}\,\vec{G}_1 = \text{Curl}\,\vec{G}_2 = 0$

$\Rightarrow \nabla U = \vec{G}_1 \quad \nabla V = \vec{G}_2$

③ make $U + iV = F(z)$ satis C-R equn's $\Rightarrow$

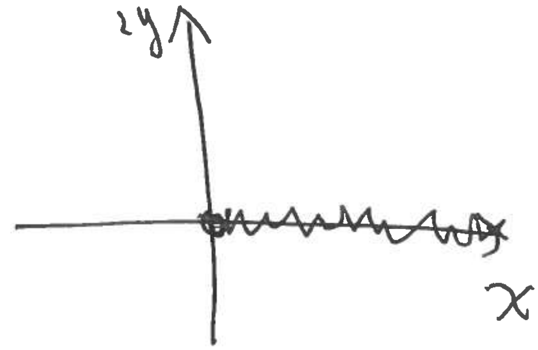
$$F'(z) = f(z)$$

$$\Rightarrow \int_C f(z)\,dz = F(B) - F(A)$$

Complex extension of FTC

<u>Note</u>: To conclude ① - ③, f must satisfy C-R in a simply connected Domain

$$f(z) = \frac{1}{z^n} \quad n \geq 1$$



satisfies CR in $z \neq 0$ <u>not simply connected</u>

If we remove +real axis, we do get s.c. region, but the anti-deriv of f (namely F) may not be continuous across branch cut.

<u>Corollary</u>: If $F'(z) = f(z)$, exists & cont (in a path connected open set like $\{z \neq 0\}$) then

$$\int_{C} f(z)\,dz = F(B) - F(A)$$

<u>Check</u>: $F(z) = U + iV$

$$F'(z) = f(z) \Rightarrow U_x = u \,,\; V_x = v \qquad (\Delta z = \Delta x)$$

$$V_y = u \,,\; U_y = -v \qquad (\Delta z = i\Delta y)$$

$$\Rightarrow \nabla U = \overrightarrow{(u,-v)} = \vec{G}_1$$

$$\nabla V = \overrightarrow{(v,u)} = \vec{G}_2$$

$$\Rightarrow \int_e \vec{G}_1 \cdot \vec{T}\, ds = \int_{t_A}^{t_B} \nabla U(x(t),y(t)) \cdot (x'(t),y'(t))\, dt$$

$$= \int_{t_A}^{t_B} \frac{d}{dt}\Big(U(x(t),y(t))\Big)\, dt = U(B) - U(A)$$

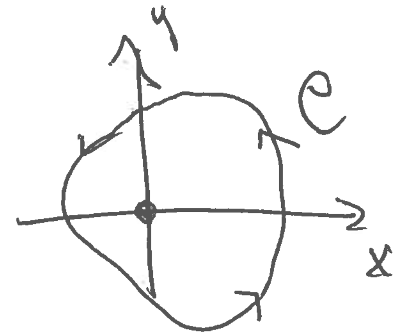
$$\int_e \vec{G}_2 \cdot \vec{T}\, ds = V(B) - V(A)$$

$$\therefore \int_e f(z)\, dz = F(B) - F(A) \checkmark$$

The problem when domain is <u>not</u> simply connected is the existence of anti-deriv $F(z)$!

<u>Theorem</u>: (Fundamental Calculation of Complex Variables)

Let C be a simple closed curve surrounding $z=0$, so $A=B$.



Then:

$$\oint_C z^n \, dz = \begin{cases} 0 & \text{if } n \neq -1 \\ 2\pi i & \text{if } n = -1. \end{cases}$$

The power $n=-1$ is <u>special</u>:

$f(z) = \frac{1}{z}$ is <u>special</u>

P.f. $n \neq -1$, $F(z) = \frac{z^{n+1}}{n+1}$ satisfies $F'(z) = z^n$.

$\therefore \oint_C z^n \, dz = F(B) - F(A) = 0$ ✓