Topology of $\mathbb{R}^2$: (Skip the proofs)

Defn. The "topology" of $\mathbb{R}^2$ is the collection of <u>open sets</u>. The subject of topology is the characterization of <u>convergence</u> in terms of the topology.

Defn: A set $\mathcal{O} \subseteq \mathbb{R}^2/\mathbb{C}$ is <u>open</u> if $\forall\, z \in \mathcal{O}\ \exists\, \varepsilon > 0$ st $B_\varepsilon(z) \subseteq \mathcal{O}$, $B_\varepsilon(z) = \{w : |w - z| < \varepsilon\}$ = "open ball radius ε center z_0"

A set $E \subseteq \mathbb{R}^2/\mathbb{C}$ is <u>closed</u> if $E = \mathcal{O}^c_{\text{open}}$

$$\mathcal{O}^c = \{z \in \mathbb{C} : z \notin \mathcal{O}\}$$

Thm: ∞-unions & finite intersection of open set are open.

∞-intersections and finite unions of closed sets are closed

Pf DeMorgan: $\left(\bigcup_\alpha E_\alpha\right)^c = \bigcap_\alpha E_\alpha^c$, $\left(\bigcap_\alpha E_\alpha\right)^c = \bigcup_\alpha E_\alpha^c$

FIP

- **Defn**: A <u>nbhd</u> of a point is any open set containing that point
  A <u>deleted nbhd</u> of z_0 is $\mathcal{O}_{open} \setminus \{z_0\}$
- Note: $\mathbb{C}$ & $\emptyset$ are the only sets both open and closed FIP
- **Defn**: $z_n \to z_0$ if $\forall \varepsilon > 0\ \exists N$ st $n > N$
  $(\lim_{n\to\infty} z_n = z_0)$ $\Rightarrow |z_n - z_0| < \varepsilon$

Defn: $\lim_{z\to z_0} f(z) = L$ if $\forall \varepsilon > 0\ \exists \delta > 0$

st if $|z - z_0| < \delta$ & $z \neq z_0$, then $|f(z) - L| < \varepsilon$

Since $|z_n - z_0| < \varepsilon \iff z_n \in B_\varepsilon(z_0)$ we say:

"$z_n \xrightarrow{n\to\infty} z_0$ if z_n is eventually inside evey nbhd of z_0"

"$f(z) \xrightarrow{z\to z_0} L$ if squeezing z within smaller and smaller deleted nbhd of z_0 forces $f(z)$ closer & closer to L"

(Note: typically $f(z)$ not defined @ $z=z_0$)

<u>Thm</u>: $\lim\limits_{z\to z_0} f(z)=L$ iff for every sequence $z_n\to z_0$ we have $f(z_n)\longrightarrow f(z_0)$. FIP

<u>Thm</u>: If a limit exists, it is unique.

<u>Thm</u>: $\lim\limits_{n\to\infty}(z_n+w_n) = \lim\limits_{n\to\infty} z_n + \lim\limits_{n\to\infty} w_n$

$$\lim_{n\to\infty}(z_n w_n) = \left(\lim_{n\to\infty} z_n\right)\left(\lim_{n\to\infty} w_n\right)$$

$$\lim_{n\to\infty}\frac{z_n}{w_n} = \frac{\lim z_n}{\lim w_n} \quad \text{if } \lim_{n\to\infty} w_n \neq 0$$

<u>Thm</u>: Same for limits of functions. FIP

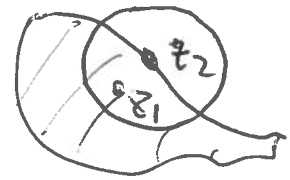
<u>Thm</u>: (Big) a sequence converges iff its <u>Cauchy</u>

<u>Defn</u>: z_n <u>Cauchy</u> if $\forall \varepsilon>0\ \exists N$ st $|z_k-z_l|<\varepsilon\ \forall k,l>N$.

- **Defn**: ∂E = "boundary of set $E \subseteq \mathbb{C}$"

Defn: $z_0 \in \partial E$ if $\forall \varepsilon$ $\exists z_1, z_2$ st

$$z_i \in B_\varepsilon(z_0), \quad z_1 \in E, \quad z_2 \in E^c$$



Thm: E closed iff $\partial E \subseteq E$

Defn: $\overline{E}$ = "closure of E" = $E \cup \partial E$

Thm: $\overline{E}$ is closed

Cor: "E is closed iff E closed under limits"
I.e. E closed iff every sequ $z_n \in E$ which converges, converges to a point in E.

- **Defn**: f is continuous at z_0 if $f(z_0)$ is defined and $\lim_{z \to z_0} f(z) = f(z_0)$

Thm: f is continuous iff the inverse image $f^{-1}(\mathcal{O})$ of every open set $\mathcal{O}$ is also open.

Thm: f cont iff $f^{-1}(E_{\text{closed}})$ is closed

Defn: $f^{-1}(\mathcal{O}) = \{z \in \mathbb{C} : f(z) \in \mathcal{O}\}$

Above assumes $f : \mathbb{C} \to \mathbb{C}$. If

$$f : A \to \mathbb{C}$$

Then still true by defining B to be open relative to A if $B = A \cap \mathcal{O}_{\text{open}}$.

• **Defn**: $E \subseteq \mathbb{C}$ is compact if every open covering of E admits a finite subcover.

Thm: (Big) $E \subseteq \mathbb{C}$ is compact if and only if E is closed and bounded iff every sequence $\{z_n\}$ in E contains a convergent subsequence $z_{n_m} \to z \in E$

Defn: $E \subseteq \mathbb{C}$ bounded if $\exists M > 0$ st

$$|z| \leq M \quad \forall z \in E$$

Thm. (Big) If a real valued function $f: E \to \mathbb{R}$ is continuous on E compact then:

(1) f is bounded on E

$(\exists M > 0 \text{ st } |f(z)| \leq M \quad \forall z \in E)$

(2) f takes on its max and min values on E

$(\exists z_1, z_2 \in E \text{ st } |f(z)| \leq |f(z_1)| \ \& \ |f(z)| \geq |f(z_2)| \ \forall z \in E)$

(if f(z), (and hence |f(z)|), is continuous)

Thm: (Big) A continuous function on a compact set E is uniformly continuous:

I.e. $f : E_{\text{compact}} \to \mathbb{C}$ cont $\Rightarrow$ f unif cont.

Defn: f uniformly continuous on E if $\forall\, \varepsilon > 0\ \exists\, \delta > 0$ s.t. $|z_2 - z_1| < \delta$ implies

$$|f(z_2) - f(z_1)| < \epsilon$$

"You can make outputs uniformly close by restricting inputs sufficiently close"

Thm (Big) If f_n are continuous and $f_n \to f$ uniformly then f is continuous

"The uniform limit of cont. fn's is cont"

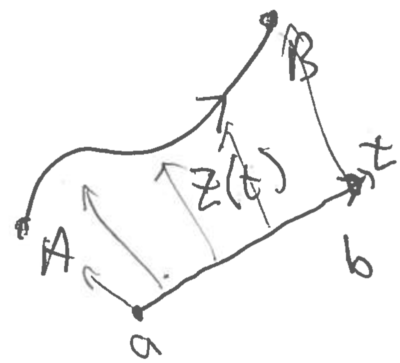
Defn: $f_n \to f$ uniformly on set E (any E) if $\forall\, \varepsilon > 0\ \exists\, N$ st $n > N$ implies $|f_n(z) - f(z)| < \varepsilon$ $\forall\, z \in E$.

Uniform convergence is the fundamental concept needed for the theory of line integrals.

Example: Let $\mathcal{C}$ be a path in $\mathbb{C}$:

$$\mathcal{C}: z(t) \quad a \le t \le b \quad z(a) = A, \ z(b) = B.$$

$z(t) = x(t) + i y(t)$



Assume $z(t)$ continuous

Prove: $\forall \varepsilon \ \exists \delta$ st if

$|t_2 - t_1| < \delta$ then $|z(t_2) - z(t_1)| < \varepsilon$

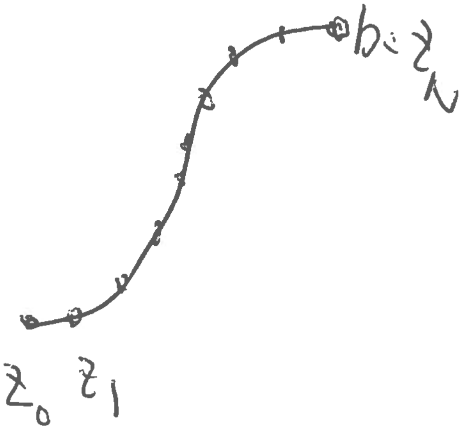
Soln: $[a,b]$ compact (closed & bounded) and $z(t)$ cont $\Rightarrow$ $z(t)$ is uniformly cont.

Thus $\forall \varepsilon \ \exists \delta$ st $|t_2 - t_1| < \delta \Rightarrow |z(t_2) - z(t_1)| < \varepsilon$

<u>Application</u>: Choose $\Delta t = \frac{b-a}{N}$ and define

$$t_k = a + k\Delta t \quad t_0 = a \quad t_N = b$$

$$z_k = z(t_k)$$



<u>Then</u> $|z_k - z_{k-1}| \longrightarrow 0$ uniformly as $N \to \infty$.

I.e. $\forall \varepsilon \; \exists \bar{N}$ st $N > \bar{N} \Rightarrow |z_k - z_{k-1}| < \varepsilon$.

Also then: If $z'(t)$ also continuous, then

$$|z_k' - z_{k-1}'| \longrightarrow 0 \text{ uniformly as } N \to \infty$$

Cor: $\int_C f(z)\,dz = \lim_{N\to\infty} \sum_{k=0}^{N} f(z_k) z_k' \Delta t$

$\int_a^b f(z(t))\, z'(t)\, dt$

The converge unif if z'

FIP (Midterm)

Claim ① If f is continuous and

$z(\cdot) \in C^1$ (i.e. both $z(t)$ & $z'(t)$ are cont)

then

$$\int_{e} f(z)\,dz = \underbrace{\int_a^b f(z(t))z'(t)\,dt}_{\substack{\text{This gives the parameterization } (x(t),y(t)) \text{ of the real} \\ \text{\& imag line integrals } \int_a^b \vec{G}_1\cdot\vec{T}\,ds \text{ \& } \int_a^b \vec{G}_2\cdot\vec{T}\,ds}} = \lim_{N\to\infty} \sum_{k=0}^{N} \underbrace{f(z_k)z_k'}_{\substack{\text{uniformly close to } f(z)z'(t) \\ \text{for } t_{k-1}\le t\le t_k}}\Delta t$$

is a well defined limit independent of how we take $\Delta t \to 0$ or where we chose $z_k = z(t_k^*)$ $t_{k-1} \le t_k^* \le t_k$

<u>Note</u>: By the same argument we needn't take t_k to be equally spaced

$$t_k = k\Delta t, \quad \Delta t = \frac{b-a}{N}$$

We can take any $t_0 < \dots < t_N$ st

$t_0 = a$, $t_N = b$ and $\max_{i=1}^{N} |t_i - t_{i-1}| = \Delta t \xrightarrow[N \to \infty]{} 0$

Claim (2) HW Prove f cont & $z(\cdot) \in C^1$

$a \le t \le b$

$z(a) = A$

$z(b) = B$

$$\Rightarrow \left| \int_a^b f(z(t))\, z'(t)\, dt \right| \le$$

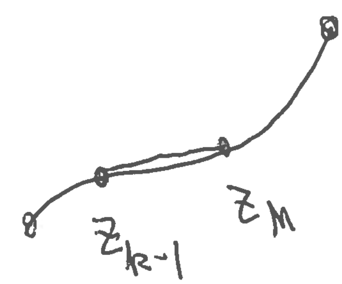
(HW) ⑩c

Claim ② : Use the same argument to prove that the length of a curve $L = \int_a^b |z'(t)|\,dt$ defined by

$$L = \int_a^b |z'(t)|\,dt \approx \lim_{N\to\infty} \sum_{k=1}^{N} |z(t_k^*) - z(t_{k-1}^*)|$$

is independent of how we choose $t_k^* \in (t_{k-1}, t_k)$

(Again, this generalizes to indept of mesh $\{t_k\}$ so long as $|\Delta t| \to 0$.)



$|z_{k-1} - z_k|$

$= \left|\frac{z_{k-1} - z_k}{\Delta t}\right| \Delta t$

$= (|z'(t_k)| + o(\Delta t))\Delta t$

Claim (3) (HW) Prove

$$\left| \int_a^b f(z(t))\, z'(t)\, dt \right| \leqslant LM$$

where $M = \underset{a \leq t \leq b}{\mathrm{Max}} |f(z(t))|$

Pf of Claim 1

Ie Choose $z_k = z(\bar{t}_k) \quad t_{k-1} \le \bar{t}_k \le t_k$

$$z_k^* = z(\bar{t}_k) \quad t_{k-1} \le \bar{t}_k^* \le t_k$$

Then

$$\left| \sum_{k=0}^{N} f(z_k) z_k' \Delta t - \sum_{k=0}^{N} f(z_k^*) (z_k^*)' \Delta t \right|$$

$$= \left| \sum_{k=0}^{N} \left(f(z_k) z_k' - f(z_k^*) (z_k^*)' \right) \Delta t \right|$$

$$\le \sum_{k=0}^{N} \underbrace{\left| f(z_k) z_k' - f(z_k^*) (z_k^*)' \right|} \Delta t$$

$f(z(t)) z'(t)$ cont on compact set $[a,b] \Rightarrow$ unif cont $\Rightarrow \forall \varepsilon \; \exists \delta$ st $|\Delta t| < \delta$ then

$$| f(z_k) z_k' - f(z_k^*) (z_k^*)' | < \varepsilon.$$

Since $\Delta t = \frac{|b-a|}{N}$, if we choose

$$N > \frac{|b-a|}{\Delta t}$$

we have

$$\sum_{k=0}^{N} \underbrace{\left| f(z_n) z_n' - f(z_n^*)(z_n^*)' \right|}_{\leq \varepsilon} \Delta t$$

$$\leq \varepsilon \sum_{k=0}^{N} \Delta t \leq \varepsilon \, |b-a| \xrightarrow[N \to \infty]{} 0 .$$

Conclude: if curves are C^1, then integrals are well defined indept of how you approx by Riemann Sums.

(Note: we don't need $\Delta t = \frac{|b-a|}{N}$, we can choose any two partitions so long as $|\Delta t| < \delta$)

This is why book restricts to curves $\mathcal{C}$ which are piecewise C^1, ie, $\mathcal{C} = \bigcup_{k=1}^{n} \mathcal{C}_k$ continuous st $\mathcal{C}_k \in \mathcal{C}_1$

(so $z(t)$ & $z'(t)$ are cont except at a finite # of connection points)