VII Applications of the Cauchy Integral Theorem

(a) Winding number + Consequences of Cauchy's Thm —

Recall —

$$\oint_{\mathcal{C}} \frac{dz}{z} = 2\pi i$$

or more generally

$$\oint_{\mathcal{C}} \frac{dz}{z - z_0} = 2\pi i$$

where $\mathcal{C}$ is a <u>simple</u> <u>closed</u> curve surrounding z_0.

Simple means it does not cross itself,

closed means $\mathcal{C}: z(t)$ $t_A \le t \le t_B$ $z(t_A) = A$ & $A = B$
$z(t_B) = B$

<u>Note</u>: $\oint_{\mathcal{C}} \frac{dz}{z - z_0} \underset{\uparrow}{=} \int_{\mathcal{C}_w} \frac{dw}{w} = 2\pi i$ ✓

$w = z - z_0$
$dw = dz$

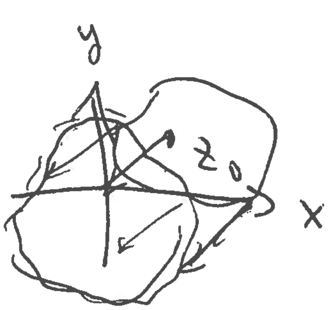
$\nwarrow$ $\mathcal{C}_w : w(t) = z(t) - z_0$

$t_A \le t \le t_B$

$w(t_A) = A - z_0$

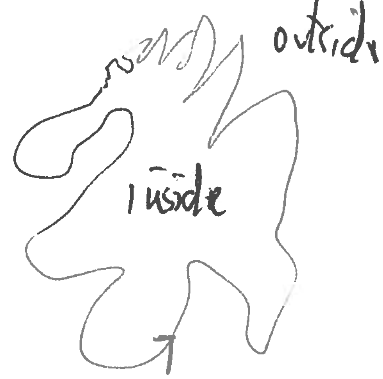
$w(t_B) = B - z_0$

$\nearrow$ shifts curve & fn by z_0



• Jordan Curve Theorem: A Continuous simple closed curve divides $\mathbb{R}^2$ into two disjt sets, one bounded & one unbounded. The bounded set is the inside, the unbounded set is the outside. We can use

$$\oint_{C} \frac{dz}{z-z_0}$$ to define the inside & outside of C:



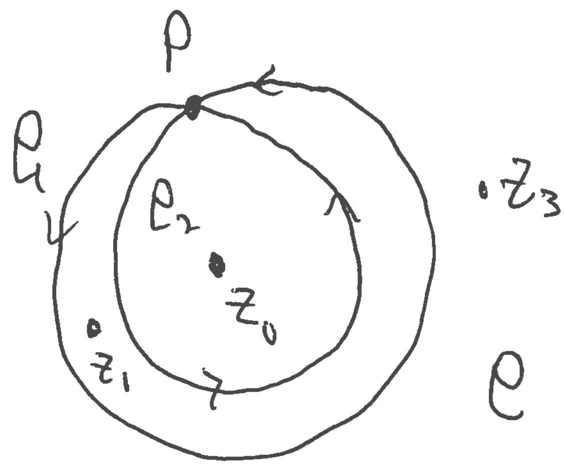
Defn: $$\frac{1}{2\pi i}\oint_{C} \frac{dz}{z-z_0} = \begin{cases} 1 & \text{if } z_0 \text{ inside} \\ 0 & \text{if } z_0 \text{ outside} \end{cases}$$

anti-clockwise $C \Rightarrow +1$

clockwise $C \Longrightarrow -1$

• More generally for non-simple closed curves, we can use $\int_{C} \frac{dz}{z-z_0} = 2\pi i$ to define the number of times the closed curve C winds around z_0.

<u>Eg</u>: If $\mathcal{C}$ winds around z_0 twice, it must cross itself, say at P. Then we can define



$$\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \quad \text{or} \quad \mathcal{C}_1 + \mathcal{C}_2$$

$$\therefore \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{dz}{z-z_0} = \frac{1}{2\pi i}\int_{\mathcal{C}_1} \frac{dz}{z-z_0} + \frac{1}{2\pi i}\int_{\mathcal{C}_2} \frac{dz}{z-z_0} = 2$$

On the other hand; ~~for general closed curve $\mathcal{C}$~~

$$\frac{1}{2\pi i}\int_{\mathcal{C}} \frac{dz}{z-z_1} = \frac{1}{2\pi i}\int_{\mathcal{C}_1} \frac{dz}{z-z_1} + \frac{1}{2\pi i}\int_{\mathcal{C}_2} \frac{dz}{z-z_1} = 1 + 0 = 1$$

& $\frac{1}{2\pi i}\int_{\mathcal{C}} \frac{dz}{z-z_3} = 0$.

$= \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{dz}{(z=z_0)}$

<u>Defn</u>: $I(\mathcal{C}; z_0)$ = winding # $\mathcal{C}$ wrt z_0 ≡ "index of z_0 wrt $\mathcal{C}$"

<u>Thm</u>: $I(\mathcal{C}; z_0) \in \mathbb{Z}$, $\mathbb{Z}^+$ for anti-clockwise curves & $\mathbb{Z}^-$ for ~~negative~~ clockwise curves.

"Proof": Deform $\mathcal{C}$ into $z_0 + re^{it}$ with $0 \le t \le 2\pi n$ if $\mathcal{C}$ winds n times around z_0 (Skip proof). We take this as Defn.

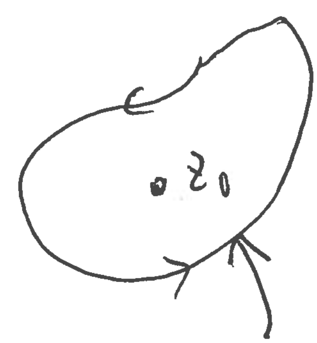
- For simple closed curves the Cauchy Integral Formula reads –

$$\oint_{\mathcal{C}} \frac{f(z)}{z - z_0}\, dz = f(z_0)\, 2\pi i$$

for any scc that surrounds z_0

Or we can represent $f(z)$ inside $\mathcal{C}$ as:

$$f(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f(w)}{w - z}\, dw$$



i.e. $\mathcal{C}: w(t)\ \ t_A \le t \le t_B\ \ w(t_A) = w(t_B)$

$$f(z) = \frac{1}{2\pi i} \int_{t_A}^{t_B} \frac{f(w(t))}{w(t) - z}\, w'(t)\, dt$$

$f(w)$

two real integrals

- More generally — for any closed curve $\mathcal{C}$ and $z \notin \mathcal{C}$, we can write

$$\int_{\mathcal{C}} \frac{f(z)}{z-z_0} = I(\mathcal{C}; z_0)\, f(z_0) \cdot 2\pi i$$

winding number — could be zero/negative $\in \mathbb{Z}$

or

$$\boxed{I(\mathcal{C}; z)\, f(z) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{f(w)}{w-z}\, dw}$$

For our purposes it is sufficient to restrict to simple closed curve $\mathcal{C}$ anti-clockwise.

Let D be the open set <u>inside</u> see $\mathcal{C}$:

$$f(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f(w)}{w-z}\, dz$$

$$\cdots \quad f^{(n)}(z) = \frac{n!}{2\pi i} \oint_{\mathcal{C}} \frac{f(w)}{(w-z)^{n+1}}\, dw$$

$$f'(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f(w)}{(w-z)^2}\, dw$$

- We can verify this by formally differentiating thru the $\int$-sign wrt z. This needs to be justified in a careful treatment. (See text)

- Consider now an open set $A \subseteq \mathbb{C}$, and assume f is analytic in A. Then $\forall\ z_0 \in A$ $\exists\ R > 0$ st $\overline{B_R(z_0)} \subseteq A$. Thus f is differentiable $\Rightarrow$ continuous in a nbhd of $\overline{B_R(z_0)}$. Since $\overline{B_R(z_0)} = \{w : |w - z_0| \leq R\}$ is closed & bded, it is compact.

Thus Real Valued f (cont on a compact set) takes on its max/min values, & $\exists\ M$ st

$$|f(z_0)| \leq M \qquad z \in \overline{B_R(z_0)}.$$

Our goal is to prove the Taylor Series for f expanded about z_0 converges in $\overline{B_R(z_0)}$

That is: $z \in \overline{B_R(z_0)}$,

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0). \qquad \text{(T-seriel)}$$

For this we need to estimate $f^{(k)}(z_0)$:

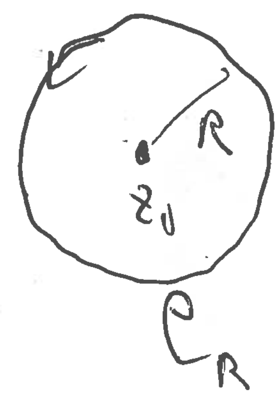
<u>Theorem</u>: (Cauchy Inequality) Assume f analytic in nbhd of $\overline{B}_R(z_0)$.

Then

$$|f^{(k)}(z_0)| \leq \frac{k!}{R^k} M$$

<u>Proof</u>: Let $\mathcal{C}$ be circle of radius R center z_0

so $\mathcal{C}_R = \partial B_R(z_0)$. Then



$$f^{(k)}(z_0) = \frac{k!}{2\pi i} \int_{\mathcal{C}_R} \frac{f(\xi)}{(\xi - z_0)^{k+1}} \, d\xi$$

Thus

$$|f^{(k)}(z_0)| \leq \frac{k!}{2\pi} \int_{\mathcal{C}_R} \frac{|f(\xi)|}{|\xi - z_0|^{k+1}}$$

$$\leq \frac{k!}{\cancel{2\pi}} \underbrace{|\mathcal{C}_R|}_{\cancel{2\pi} R} M \cdot \frac{1}{R^{\cancel{k+1}}} = \frac{k!}{R^k} M$$

Main Estimate FIP
$$\Big|\int_{\mathcal{C}} f(z)\,dz\Big| \leq |\mathcal{C}|\, M$$
$|\mathcal{C}|$ = length of $\mathcal{C}$, M = max of $|f|$ on $\mathcal{C}$

<u>Cor</u> 1 (Liouvilles Thm – Big) Any bounded entire function is constant.

<u>Defn</u>: f is entire if it is analytic in all of $\mathbb{C}$.

<u>Proof</u>: Assume f is entire and $|f(z)| \le M$ for all $z \in \mathbb{C}$. Then $z \in B_R(z) \Rightarrow$

$$|f'(z)| \le \frac{1}{R} M \quad \cancel{\text{for all } z}.$$

Since we can take R arbitrarily large,

$$|f'(z)| = 0 \quad \forall z \in \mathbb{C} \Rightarrow f'(z) = 0.$$

$\therefore f(z) = \text{const}$ ✓

Cor 2 (Fundamental Thm of Algebra)

Let $P(z)$ be any complex polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n, \quad a_k \in \mathbb{C}, \quad a_n \neq 0$$

Then $P(z)$ has at least one root $z_0 \in \mathbb{C}$ where $P(z_0) = 0$. ($\Rightarrow P(z) = (z - z_0)\, q(z)$ $\deg q = n-1$
$\Rightarrow P(z) = \prod_{k=1}^{n} c(z - z_k) \Rightarrow$ every poly factors in $\mathbb{C}$)

Proof: Assume $\nexists\, z_0$ st ~~$P(z_0)$~~ $P(z_0) = 0$. Then $\frac{1}{P(z)}$ is an entire function. We prove $\frac{1}{P(z)}$ would then be bounded, and hence constant by Cor 1. This is a ※ so such a z_0 with $P(z_0) = 0$ must exist.

- To prove $\frac{1}{P(z_0)}$ is bounded if it is entire, note that

$\lim_{z \to \infty} P(z) = \infty$. That is, for $z = re^{i\theta}$, $r = |z|$, as $r \to \infty$ the $a_n z^n$ term wins. Thus $\exists\, R > 0$ st if $|z| > R$, then $|P(z)| > 1$. In this case

$$\left|\frac{1}{P(z)}\right| = \frac{1}{|P(z)|} \leq 1 \quad \text{for } |z| \cancel{\geq} R$$

But $P(z)^{-1}$ cont on compact set $\overline{B_R(0)}$ $\Rightarrow$ $|P(z)^{-1}| \leq M$ ~~on~~ for $|z| \leq R$. $\Rightarrow \left|\frac{1}{P(z)}\right| \leq \begin{cases} 1 & r \geq R \\ M & r \leq R \end{cases}$

$\therefore$ $\frac{1}{P(z_0)}$ is bounded entire $\Rightarrow$ const $\times$

$\therefore$ $P(z_0) = 0$ some $z_0 \in \mathbb{C}$.

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More carefully: HW

Lemma: $\exists R > 0$ st $|z| \geq R \Rightarrow |P(z)| \geq 1$.

Proof Let $z = re^{i\theta}$, $|z| = r > 1$. Then

$$|P(z)| = |a_0 + \cdots + a_n z^n| \geq |a_n z^n| - \underbrace{|a_0| - |a_1| r - \cdots - |a_{n-1}| r^{n-1}}_{n \text{ terms},\ r>1}$$

$$\geq |a_n| r^n - n a r^{n-1}$$

$a = |a_0| + \cdots + |a_{n-1}|$

$$\geq \underbrace{(|a_n| r - na)}_{\geq 1 \text{ for } r > \frac{na}{|a_n|}} r^{n-1}$$

Morera's Thm: If f is continuous in an open set $D \subseteq \mathbb{C}$, and $\oint_{\mathcal{C}} f(z)\,dz = 0 \ \forall\, \mathcal{C} \in D$, then f is analytic in D.

Proof: $\oint_{\mathcal{C}} f(z)\,dz = 0 \Rightarrow \int_{\mathcal{C}} f(z)\,dz$ is indept of path $\mathcal{C}$ from A to B, & we can ~~write~~ define

$$F(z) = \int_A^z f(\xi)\,d\xi,$$ which is the same for any path in D from A to z.

Claim: $F'(z) = f(z)$. For this we prove

$$\left| \frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) \right| = \left| \frac{1}{\Delta z} \int_z^{z+\Delta z} f(\xi)\,d\xi - f(z) \right|$$

$$= \left| \frac{1}{\Delta z} \int_z^{z+\Delta z} f(\xi)\,dz - f(z) \frac{1}{\Delta z} \int_z^{z+\Delta z} d\xi \right|$$

$$= \left| \frac{1}{\Delta z} \int_z^{z+\Delta z} f(\xi) - f(z)\,d\xi \right| \le \frac{1}{|\Delta z|} \int_z^{z+\Delta z} |f(\xi) - f(z)|\,dz$$

$$\le \frac{1}{|\Delta z|} \cdot |\Delta z| \max_{\xi \in B_r(z)} |f(\xi) - f(z)| \xrightarrow[\Delta z \to 0]{} 0 \Rightarrow F' = f$$

analytic

<u>Cor.</u> If f is continuous in open set A and differentiable except at $z_0 \in A$, then f is analytic at z_0.

<u>Proof</u>: f cont $\Rightarrow |f(z)| \leq M < \infty$ for $z \in \overline{B_\rho(z_0)} \subset A$.

$$\int_{\mathcal{C}} f(z)\,dz = \int_{\mathcal{C}_\varepsilon} f(z)\,dz$$

$$\left| \int_{\mathcal{C}_\varepsilon} f(z)\,dz \right| \leq \underbrace{|\mathcal{C}_\varepsilon|}_{2\pi\varepsilon} M \xrightarrow[\varepsilon \to 0]{} 0$$

$\therefore \int_{\mathcal{C}} f(z)\,dz = 0 \quad \forall$ closed $\mathcal{C} \Rightarrow f$ analytic

Maximum Principle: Assume f analytic in A_{open}

Cauchy Thm: $$\frac{1}{2\pi i}\oint_{C} \frac{f(z)}{z-z_0}\,dz = f(z_0) \qquad C \subseteq A$$

Take $C \equiv C_R$ $\quad z(t) = z_0 + Re^{it}$

$0 \le t \le 2\pi$

$$f(z_0) = \frac{1}{2\pi \cancel{i}} \int_0^{2\pi} \frac{f(z_0 + Re^{it})}{\cancel{Re^{it}}} \cancel{iRe^{it}}\, dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{it})\, dt =$$ "Average of $f(w)$ on circle of radius R center z_0"

Since "the average is always ~~smaller than~~ lies between the the max or min values" (must prove this in a complex sense)

Thm: An non-constant analytic function in A always takes its max & min modulus on ∂A.

- Harmonic fn's in $\mathbb{R}^2$ —

Note: $f(z) = u(x,y) + i\,v(x,y)$ analytic

$\Rightarrow$ CR $\quad u_x = v_y \quad u_y = -v_x \quad \Rightarrow \quad u_{xx} = v_{yx} \quad v_{yy} = u_{xy}$

$u_{yy} = -v_{xy} \quad v_{xx} = -u_{xy}$

$u_{xx} + u_{yy} = 0 \quad v_{xx} + v_{yy} = 0$

Defn: $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ = Laplacian.

A function u st $\Delta u = 0$ is harmonic

Conclude: the real and imaginary parts of an analytic function are harmonic fn.

If $\Delta u = 0$ & $\Delta v = 0$ & $u + iv$ is analytic, we say u & v are harmonic conjugates.

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Now: $f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z+re^{it})\,dt$

Since $f(z) = u + iv$, $u_0 + iv_0 = u(x_0,y_0) + iv(x_0,y_0)$

$$u_0 + iv_0 = \frac{1}{2\pi}\int_0^{2\pi}(u+iv)\,dt \qquad \underset{\sim}{x}_0 = (x_0,y_0)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} u(\underset{\sim}{x}_0+re^{it})\,dt + \frac{i}{2\pi}\int_0^{2\pi} v(\underset{\sim}{x}_0+re^{it})\,dt$$

$$u(x_0,y_0) = \frac{1}{2\pi}\int_0^{2\pi} u(\underset{\sim}{x}_0+re^{it})\,dt$$

$$v(x_0,y_0) = \frac{1}{2\pi}\int_0^{2\pi} v(\underset{\sim}{x}_0+re^{it})\,dt$$

"Harmonic functions = their average on the ball of radius R" in the Real sense.

HW: The average is $\leq$ Max & $\geq$ Min, so Harmonic functions take their max/min values on the $\partial A \Leftrightarrow$ No interior Max/Min points.

• To fill in details – we need to

(1) show every C^2 harmonic u has a harmonic conjugate v s.t. $u+iv$ analytic. (omit). Then $\Delta u = 0 \Rightarrow u$ takes max/min on ∂A (HW)

(2) we need to prove $|f(z)|$ takes its max on ∂A. (omit - in text)

Proof of (1): Assume $\Delta u = 0$. We find v s.t. $u+iv$ is analytic. But only need C-R equns.

$u_x = v_y$
$u_y = -v_x$

so define $g(z) = u_x - i u_y = U + iV$.

We prove g analytic $\Rightarrow$ g has antideriv $\hat{u} + i\hat{v}$ analytic for some $\hat{v}$. But $U_x = u_{xx} = -u_{yy} = V_y$ and $U_y = u_{xy} = u_{yx} = -V_x$ $\Rightarrow$ g analytic $\Rightarrow$ g has anti-deriv on s.c. domain. $\cancel{g(z) = \hat{u} + i\hat{v}}$. ~~But~~

$\cancel{\hat{u}_x = U = u_x}$, $\cancel{\hat{v}_x = -u_y = \hat{v}}$

Let $f(z) = \hat{u} + i\hat{v}$ denote the anti-deriv of g

So $f'(z) = g(z) = u_x - i u_y$

Also by def deriv: $f'(z) = \hat{u}_x + i\hat{v}_x$ $\Big\}$ $u_x = \hat{u}_x$, $u_y = -\hat{v}_x$

C-R for $f \Rightarrow \hat{u}_y = -\hat{v}_x = u_y$

$\therefore \nabla\hat{u} = \nabla u \Rightarrow u = \hat{u} + \text{Const}.$

Thus $\hat{v}$ is a harmonic conjugate of u since

$f(z) = u + i\hat{v} - \text{Const}$ is analytic ✓

<u>Cor</u>: If $\Delta u = 0$ and $u \in C^2$ then $u \in C^\infty$.
(In fact u has power series expansion about every point!)

<u>Conclude</u>: A PDE regularity result for solutions of $\Delta u = 0$ follows from complex variables!