Name: ______________________________

Student ID#: ________________________

Section: ________________________

Final Exam

Thursday March 21, 1:00-3:00pm

MAT 185A, Temple, Winter 2019

Print name and ID's clearly. Have student ID ready. Write solutions clearly and legibly. Do not write near the edge of the paper or the stapled corner. Correct answers with no supporting work will not receive full credit. No calculators, notes, books, cellphones...allowed.

Problem	Your Score	Maximum Score
1		20
2		20
3		20
4		20
5		20
6		20
7		20
8		20
9		20
10		20
Total		200

Problem #1 (20pts): Recall that $\cos z = \frac{e^{iz}+e^{-iz}}{2}$, for $z = x + iy \in \mathcal{C}$.

(a) Show that $\cos z$ reduces to $\cos x$ when $z = x \in \mathcal{R}$.

Solution: $e^{ix} = \cos x + i \sin x$, so

$$\frac{e^{ix} + e^{-ix}}{2} = \frac{\cos x + i \sin x + \cos(-x) + i \sin(-x)}{2} = \frac{2\cos x}{2} = \cos x$$

(b) Find $u(x, y)$ and $v(x, y)$ real so that $\cos(z) = u(x, y) + iv(x, y)$.

Solution:

$$\begin{aligned} \cos(x+iy) &= \frac{e^{ix-y} + e^{-ix+y}}{2} = \frac{e^{-y}(\cos x + i \sin x) + e^{y}(\cos(-x) + i \sin(-x))}{2} \\ &= \frac{e^{y} + e^{-y}}{2} \cos x - i \frac{e^{y} - e^{-y}}{2} \sin x \end{aligned}$$

so

$$u = \cosh(y)\cos(x), \quad v = -\sinh(y)\sin(x)$$

(c) Prove $f(z) = \sin z$ satisfies the Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$.

Solution:

$$\begin{aligned} u_x &= -\cosh(y)\sin(x) = v_y = -\cosh(y)\sin(x) \\ u_y &= \sinh(y)\cos(x) = -v_x = \sinh(y)\cos(x) \end{aligned}$$

Problem #2 (20pts): Let $z = 3i$. Find $z^{1/13}$. (That is, find all complex numbers w such that $w^{13} = z$.)

Solution: Let $z = re^{i\theta}$ with $r = 3$, and let $n = 13$.

$$z^{1/n} = e^{\frac{1}{n}\log(z)} = e^{\frac{1}{n}\log(r) + i\frac{\pi/2 + 2k\pi}{n}} = r^{1/n} e^{i\frac{\pi/2+2k\pi}{13}}$$

for $k = 0, ..., 12$. The n angles are

$$\theta_k = \frac{\pi/2 + 2k\pi}{13} = \frac{1+4k}{13}\frac{\pi}{2}.$$

Problem #3 (20pts):

(a) Assume that f^{-1} and f are inverses of each other, and $w = f^{-1}(z)$. Prove that

$$\frac{d}{dz} f^{-1}(z) = \frac{1}{\frac{d}{dw} f(w)}.$$

Solution: Since $f(f^{-1}(z)) = z$, differentiating both sides and using the Chain Rule gives

$$f'(f^{-1}(z)) \frac{d}{dz} f^{-1}(z) = 1.$$

Thus $\frac{d}{dz} f^{-1}(z) = 1/f'(f^{-1}(z)) = 1/f'(w)$.

(b) The *logarithm* is defined as the inverse of the exponential, so $w = log(z)$ if and only if $z = e^w$. Use part **(a)** together with properties of the exponential to derive $\frac{d}{dz} \log z$.

Solution: $\frac{d}{dz} \log z = \frac{1}{\frac{d}{dw} e^w} = \frac{1}{e^w} = 1/z$.

Problem #4 (20pts): Assume $f(z) = u + iv$ is analytic in an open set containing the closure of the ball $B_R(z_0)$, and let C_R denote the positively oriented closed curve which is its boundary.

(a) Prove that u at the center is given by its average value, i.e., prove

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + Re^{it}) dt.$$

Solution: By the Cauchy integral formula,

$$f(z_0) = \frac{1}{2\pi i} \int_{C_R} \frac{f(w)}{w - z_0} dw.$$

Using $z(t) = z_0 + Re^t$, $0 \leq t \leq 2\pi$, $dz = iRe^{it}dt$ we have

$$f(z_0) = \frac{1}{2\pi i} \int_{C_R} \frac{f(z_0 + Re^{it})}{w - z_0} dw = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{it}) dt.$$

Taking the real part gives the desired answer.

Problem #5 (20pts): Assume only that f is continuous, but that for any points A, B in the complex plane, $\int_C f(z)dz$ is independent of path C taking A to B. Let point A be fixed. Prove: $F(z) = \int_A^z f(z)dz$ is an anti-derivative of f. (Here $\int_A^z$ denotes the integral along any path from A to z.)

Solution: Since integration can be along any path,

$$\begin{aligned}
\left|\frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z)\right| &= \left|\frac{1}{\Delta z}\left(\int_z^{z+\Delta z} f(w)dw\right) - f(z) frac1\Delta z \int_z^{z+\Delta z} dw\right| \\
&= \left|\frac{1}{\Delta z}\left(\int_z^{z+\Delta z} f(w) - f(z)dw\right)\right| \\
&\leq \frac{1}{|\Delta z|}\int_z^{z+\Delta z} |f(w)-f(z)||dw| \\
&\leq \frac{1}{|\Delta z|}|\Delta z| Max_{|w-z|\leq|\Delta z|}|f(w)-f(z)| \to 0 \quad as \quad \Delta z \to 0
\end{aligned}$$

because f is continuous at z. Thus $F'(z) = f(z)$ by definition.

Problem #6 (20pts): Recall that Cauchy's Inequality states that if f is analytic in a neighborhood of $\overline{B_R(z_0)}$, then $|f^{(k)}(z_0)| \leq \frac{k!}{R^k} M$, where M is the maximum value of f in $\overline{B_R(z_0)}$. (Here $B_R(z_0)$ denotes the open ball with center z_0 and radius R, and the bar on top denotes its closure.)

(a) Use Cauchy's Inequality to prove Liouville's Theorem, that every bounded entire function is constant.

Solution: Assume $f(z)$ is bounded and entire. Bounded means there exists $M > 0$ such that $f(z) \leq M$ for all $z \in \mathcal{C}$. Entire means we can apply Cauchy's Inequality to $B_R(z)$ for any $R > 0$, any z. In the case $k = 1$ this is

$$|f'(z)| \leq \frac{1}{R^k} M \to 0 \quad as \quad R \to \infty.$$

Thus $f'(z) = 0$ for all z, implying $f(z) = const.$

(b) Use Liouville's Theorem to prove that every polynomial $P(z)$ of order $n \geq 1$ has a complex root. (You may assume that every polynomial $P(z)$ is non-constant and $lim_{z\to\infty} P(z) = \infty$ when $n \geq 1$.)

Solution: Assume $P(z)$ is a polynomial and $P(z) \neq 0$ for all $z \in \mathcal{C}$. Then $1/P(z)$ is analytic. Since $lim_{z\to\infty} P(z) = \infty$, there exists an $R > 0$ such that $P(z) \geq 1$ for $|z| > R$. Thus $|1/P(z)| \leq 1$ for all $|z| \geq R$ But $1/P(z)$ is continuous and hence bounded on the closed ball $|z| \leq R$, so there exists M' such that $|P(z)| \leq M'$ for $|z| \leq R$. It follows that $|1/P(z)| \leq 1 + M'$, so $1/P(z)$ is a bounded entire function, and hence constant by Liouville's Theorem. Conclude that if $P(z)$ is not constant, then $P(z)$ must have a root.

Problem #7 (20pts): Assume f is analytic everywhere except for a singularities at $z = \pm 2i$.

(**a**) Recall the Taylor series $f(z) = \sum_{k=0}^{\infty} c_k(z - z_0)^k$. Assuming you can differentiate the series term by term, derive a formula for the c_k in terms of the value of f and its derivatives at $z = z_0 \neq \pm 2i$.

Solution:

$$\frac{d^k}{dz^k} f(z_0) = \sum_{k=0}^{\infty} \frac{d^k}{dz^k} c_k(z - z_0)^k|_{z=z_0} = k!c_k.$$

so

$$c_k = \frac{f^k(z_0)}{k!}.$$

(**b**) Give the radius of convergence of the Taylor series at $z_0 \neq \pm 2i$, [Correction add $\pm$] and state for what radii the Taylor series converges, converges uniformly, diverges, and for which radii it may or may not converge. (No proofs required.)

Solution: Radius of convergence R=distance to the nearest singularity of f so $R = Min\{|z - 2i|, |z + 2i|\}$. Convergence for $|z = z_0| < R$, divergence for $|z - z_0| > R$, and uniform convergence for $|z - z_0| < r < R$. Conditional convergence at $|z - z_0| = R$.

(**c**) Recall the Laruent series $f(z) = \sum_{k=1}^{\infty} \frac{c_{-k}}{(z-z_0)^k} + \sum_0^{\infty} c_k(z-z_0)^k$. Determine the annulus of convergence of the Laurent series at $z_0 = 2i$. State for which annuli the Laurent series converges, and state for which annuli it converges uniformly. (No proofs required.)

Solution: Solution: Convergence in the largest annulus centered at $z_0 = 2i$ which is singularity free. Thus convergence is in $0 < |z - 2i| < 4$, divergence for $|z - 2i| > 4$ and $z = 2i$. Uniform convergence on compact subsets of $0 < |z - 2i| < 4$.

Problem #8 (20pts):(**a**) Find the residues of the function $f(z) = \frac{1}{(z-2i)(z+i)^2}$ at $z = 2i$ and $z = -i$.

Solution: $z = 2i$ is a simple pole, $z = -i$ is a double pole.

$$R(f; 2i) = \lim_{z \to 2i} f(z)(z - 2i) = \frac{1}{(2i + i)^2} = -\frac{1}{9}.$$

$$R(f; -i) = \lim_{z \to -i} \frac{d}{dz} \left\{ f(z)(z + i)^2 \right\} = -\frac{1}{(-i - 2i)^2} = \frac{1}{9}.$$

(**b**) Find the $\int_{\mathcal{C}} f(z)dz$ where $\mathcal{C}$ is the positively oriented circle of radius $r = 3/2$ centered at the origin.

Solution: $\int_{\mathcal{C}} f(z)dz = 2\pi i R(f; -i) = \frac{2\pi i}{9}$.

Problem #9 (20pts): Evaluate $\int_{-\infty}^{+\infty} \frac{1}{2+z^2}\, dz$ by the method in class.

Solution: $f(z) = \frac{1}{2+z^2} = \frac{1}{(z-\sqrt{2}i)(z+\sqrt{2}i)}$. Let $\mathcal{C} = \mathcal{C}_R + \mathcal{C}_{[-R,R]}$ where $\mathcal{C}R$ is the half circle of radius R center zero above the x-axis, and $\mathcal{C}_{[-R,R]}$ is on the real axis. Then

$$\int_{\mathcal{C}} f(z)dz = 2\pi i R(f;\sqrt{2}i) = 2\pi i \frac{1}{\sqrt{2}i + \sqrt{2}i} = \frac{\pi}{\sqrt{2}}.$$

But

$$\left| \int_{\mathcal{C}_R} f(z)dz \right| \leq |\mathcal{C}_R| \, Max_{z\in\mathcal{C}_R} |f(z)| \leq \frac{\pi R}{R^2 - 2} \to 0 \;\text{ as }\; R \to \infty.$$

Thus

$$\lim_{R\to\infty} \int_{\mathcal{C}} f(z)dz = \lim_{R\to\infty} \int_{\mathcal{C}_{[-R,R]}} f(z)dz = \int_{-\infty}^{+\infty} f(x)dx = \frac{\pi}{\sqrt{2}}.$$

Problem #10 (20pts): Evaluate $\int_0^{2\pi} \frac{1}{2+\sin t}\, dt$ by the method in class.

Solution: Let $I = \int_0^{2\pi} \frac{1}{2+\sin t}\, dt$. View this as a parameterization of a complex integral around the unit circle, so

$$z(t) = e^{it}, \quad dz = ie^{it}dt = izdt, \quad \sin t = \frac{e^{it} - e^{-it}}{2i} = \frac{z - 1/z}{2i}.$$

Substituting gives

$$I = \int_{\mathcal{C}_0} \frac{1}{2 + \left(\frac{z-1/z}{2i}\right)} \frac{dz}{iz} = \int_{\mathcal{C}_0} \frac{2dz}{z^2 + 4iz - 1}.$$

By quadratic formula, the roots of the denominator are

$$z_{\pm} = \frac{-4i \pm \sqrt{-16+4}}{2} = \frac{-4i \pm 2\sqrt{3}i}{2} = (-2 \pm \sqrt{3})i.$$

Thus z_+ lies inside the unit circle, so the residue theorem gives

$$I = 2\pi i Res(f; (-2+\sqrt{3})i) = 2\pi i \frac{2}{z_+ - z_-} = \frac{4\pi i}{2\sqrt{3}i} = \frac{2\pi}{\sqrt{3}}.$$