Name: ______________________________

Student ID#: ________________________

Section: ___________________________

Midterm Exam 1

Friday, February 2

MAT 185A, Temple, Winter 2024

Print names and ID's clearly, and have your student ID ready to be checked when you turn in your exam. Write the solutions clearly and legibly. Do not write near the edge of the paper or the stapled corner. Show your work on every problem. Correct answers with no supporting work will not receive full credit. Be organized and use notation appropriately. No calculators, notes, books, cellphones, etc. may be used on this exam.

Problem	Your Score	Maximum Score
1		20
2		20
3		20
4		20
5		20
Total		100

Problem #1 (20pts): Recall $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$.

(a) Prove that $\frac{d}{dz}\sin z = \cos z$.

$$\frac{d}{dz}\frac{e^{iz}-e^{-iz}}{2i} = \frac{\cancel{i}e^{iz}-(-\cancel{i})e^{-iz}}{2\cancel{i}} = \frac{e^{iz}+e^{-iz}}{2} = \cos z$$

(b) Find the real and imaginary parts of $f(z) = \sin z$, and verify directly they satisfy the Cauchy Riemann equations $u_x = v_y$, $u_y = -v_x$.

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i}$$

$$= \frac{1}{2i}\left\{ e^{-y} e^{ix} - e^{y} e^{-ix} \right\}$$

$$= \frac{1}{2i}\left\{ e^{-y}(\cos x + i \sin x) - e^{y}(\cos(-x) + i \sin(-x)) \right\}$$

$$= \frac{1}{2i}\left\{ e^{-y} \cos x - e^{y} \cos x + i e^{-y} \sin x + i e^{y} \sin x \right\}$$

$$= \frac{1}{2i}\left\{ -(e^{y} - e^{-y}) \cos x + i(e^{y} + e^{-y}) \sin x \right\}$$

$$= \underbrace{i \frac{e^{y} - e^{-y}}{2} \cos x}_{u} + \underbrace{\frac{e^{y} + e^{-y}}{2} \sin x}_{v}$$

$$u_x = -\frac{e^{y} - e^{-y}}{2} \sin x = v_y, \quad u_y = \frac{e^{y} + e^{-y}}{2} \cos x = -v_x$$

Problem #2 (20pts): The complex conjugate of $z = x + iy$ is $\bar{z} = x - iy$.

(a) Prove that $\bar{z}\bar{w} = \overline{zw}$ for all $z, w \in \mathbb{C}$.

$$\bar{z}\bar{w} = (x - iy)(u - iv) = xu - yv + i(-uy - xv)$$

$$\overline{zw} = \overline{(x+iy)(u+iv)} = \overline{xu - yv + i(yu + xv)}$$

$$= xu - yv - i(yu + xv) \checkmark$$

(b) Let $f(z) = \frac{\bar{z}}{x^2+y^2}$. Prove that $f(z)$ is analytic, or else prove that it is not analytic, for $z \neq 0$.

$$f(z) = \frac{\bar{z}}{z\bar{z}} = \frac{1}{z} \quad \text{analytic since } \frac{d}{dz}\frac{1}{z} = -\frac{1}{z^2}$$

(c) Let $f(z) = (\bar{z})^2$. Prove that $f(z)$ is entire, or else prove that it is not entire.

$$f(z) = (\bar{z})^2 = (x - iy)^2 = \underbrace{x^2 - y^2}_{u} \underbrace{- 2xy}_{v}\, i$$

$$u_x = 2x \qquad v_y = -2x \qquad u_x \neq v_y \Rightarrow \text{not analytic}$$

Problem #3 (20pts): Recall the Math 21D notation $\vec{\mathbf{G}} = \overrightarrow{(M, N)}$ for a vector field, assume $f(z) = u + iv$ is analytic (i.e. complex differentiable), and let $dz = dx + idy$.

(a) Use Leibniz calculus of differentials to derive vector fields $\vec{\mathbf{G}}_1$ and $\vec{\mathbf{G}}_2$ such that

$$\int_C f(z)dz = \int_C \vec{\mathbf{G}}_1 \cdot \vec{\mathbf{T}} ds + i \int_C \vec{\mathbf{G}}_2 \cdot \vec{\mathbf{T}} ds.$$

$$\int_C f(z)\,dz = \int_C (u+iv)(dx+idy)$$

$$= \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy$$

$$\vec{G}_1 = \overrightarrow{(u,-v)} \qquad \vec{G}_2 = \overrightarrow{(v,u)}$$

$$= \int_C \vec{G}_1 \cdot \vec{T}\,ds + i\int_C \vec{G}_2 \cdot \vec{T}\,ds$$

(b) Assume $\nabla U(x,y) = \vec{\mathbf{G}}_1$ and $\nabla V(x,y) = \vec{\mathbf{G}}_2$. Prove that $F(z) = U + iV$ is differentiable, and $F'(z) = f(z)$.

$\vec{G}_1 = \overrightarrow{(u, -v)} \Rightarrow U_x = u, \; U_y = -v$

$\vec{G}_2 = \overrightarrow{(v, u)} \Rightarrow V_x = v, \; V_y = u$

$\Rightarrow U_x = u = V_y, \; U_y = -v = -V_x$

$\Rightarrow U + iV$ satisfies CR.

$F(z) = U + iV$ is analytic $\Rightarrow F'(z)$ exists

$F'(z) = U_x + iV_x = u + iv = f(z)$ ✓

$\uparrow$

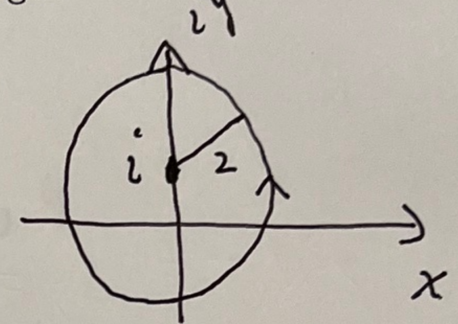
$\Delta z = \Delta x$

Problem #4 (20pts): Let C denote the positively oriented circle of center $z_0 = i$ and radius $r = 2$. Evaluate the line integral

$$\int_C \left\{ z^n + \frac{1}{z-i} \right\} dz$$

by *direct parameterization*, where n is any positive integer.

Let $z(t) = 2e^{it} + i \qquad 0 \le t \le 2\pi$



$z'(t) = 2ie^{it}\,dt$

$$\int_C z^n + \frac{1}{z-i}\, dt = \int_C z(t)^n z'(t)\, dt + \int_C \frac{2ie^{it}}{2e^{it}}\, dt$$

$$= \int_C \frac{d}{dt}\left(\frac{z(t)^{n+1}}{n+1}\right) dt + \int_C i\, dt$$

$$= \cancel{\frac{z^{n+1}(2\pi)}{n+1}} - \cancel{\frac{z^{n+1}(0)}{n+1}} + 2\pi i = 2\pi i$$

Problem #5 (20pts):

(a) Find all $w \in \mathbb{C}$ such that $w = i^{1/7}$.

Find w st $w = i = e^{i\pi/2}$

$$\boxed{w_0 = e^{i\pi/10}}, \quad \Delta\theta = \frac{2\pi}{5}$$

$$w_k = e^{i\left(\frac{\pi}{10} + k\Delta\theta\right)} \qquad k = 0, \dots, 4$$

$$= e^{i\left(\frac{\pi}{10} + \frac{2k\pi}{5}\right)}$$

$$\Rightarrow \boxed{\begin{aligned} w_1 &= e^{i\left(\frac{\pi}{10} + \frac{2\pi}{5}\right)} \\ w_2 &= e^{i\left(\pi/10 + \frac{4\pi}{5}\right)} \\ w_3 &= e^{i\left(\pi/10 + \pi\right)} \\ w_4 &= e^{i\left(\frac{\pi}{10} + \frac{8\pi}{5}\right)} \end{aligned}}$$

(b) Find all $w \in \mathbb{C}$ such that $w = i^{\pi+ie}$.

$W = z^a = e^{a \log z}$ $\quad a = \pi + ie, \quad z = i = e^{i\pi/2}$

$\log i = i\left(\frac{\pi}{2} + 2k\pi\right)$

$W_k = e^{(\pi+ie)\left(i\left(\frac{\pi}{2}+2k\pi\right)\right)}$ $\quad k \in \mathbb{Z}$

$= e^{i\pi\left(\frac{\pi}{2}+2k\pi\right)} e^{-e\left(\frac{\pi}{2}+2k\pi\right)}$

$\Rightarrow \quad \theta_k = \frac{\pi^2}{2} + 2k\pi^2, \quad r_k = e^{-\left(\frac{e\pi}{2}+2k\pi e\right)}$

Satisfies $W_k = r_k e^{i\theta_k} = i^{\pi+ie}$