Name: ____________________

Student ID#: ____________________

Section: ____________________

Midterm Exam 1

Friday, February 1

MAT 185A, Temple, Winter 2019

Print names and ID's clearly, and have your student ID ready to be checked when you turn in your exam. Write the solutions clearly and legibly. Do not write near the edge of the paper or the stapled corner. Show your work on every problem. Correct answers with no supporting work will not receive full credit. Be organized and use notation appropriately. No calculators, notes, books, cellphones, etc. may be used on this exam.

Problem	Your Score	Maximum Score
1		20
2		20
3		20
4		20
5		20
Total		100

Problem #1 (20pts): Recall that $e^{i\theta} = \cos\theta + i\sin\theta$. Prove that

$$e^{i(\theta_1+\theta_2)} = e^{i\theta_1}e^{i\theta_2}$$

.

Solution:

$$\begin{aligned} e^{i(\theta_1+\theta_2)} &= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)) \\ &= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i\sin\theta_2\cos\theta_1 + i\sin\theta_1\cos\theta_2 \end{aligned}$$

On the other hand

$$\begin{aligned} e^{i\theta_1}e^{i\theta_2} &= (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i\sin\theta_2\cos\theta_1 + i\sin\theta_1\cos\theta_2. \end{aligned}$$

SO the two are equal.

Problem #2 (20pts): Let $\vec{G} = \overrightarrow{(M, N)}$ be a vector field in the plane, where $M = M(x, y)$, $N = N(x, y)$ are real valued functions of (x, y). Let C be a smooth curve in the plane taking point A to point B. Use Leibniz's substitution principle to show the following are equal: (Here $\mathbf{r}(t) = \overrightarrow{(x(t), y(t))}$ denotes any smooth parameterization of curve C.)

$$\int_C \overrightarrow{G} \cdot \overrightarrow{T}\ ds = \int_C M dx + N dy = \int_C \overrightarrow{G} \cdot d\overrightarrow{\mathbf{r}} = \int_C \overrightarrow{G} \cdot \overrightarrow{\mathbf{v}}\ dt.$$

Solution: Given $\mathbf{r}(t)$ we have

$$\frac{d\mathbf{r}}{dt} = \mathbf{v} = \frac{ds}{dt}\mathbf{T}.$$

Thus

$$\mathbf{T}ds = \mathbf{v}dt = d\mathbf{r} = \overrightarrow{(dx, dy)}$$

So

$$\int_C \overrightarrow{G} \cdot \overrightarrow{T}\ ds = \int_C \overrightarrow{G} \cdot \overrightarrow{\mathbf{v}}\ dt = \int_C \overrightarrow{G} \cdot d\overrightarrow{\mathbf{r}} = \int_C M dx + N dy.$$

Problem #3 (20pts): Prove that $f(z) = 1/z$ is analytic (its complex derivative exists) for all $z = x + iy \neq 0$ two ways:

(1) By showing directly $\lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z}$ exists.

(2) By proving $1/z = u(x,y)+iv(x,y)$ satisfies the Cauchy-Riemann equation $u_x = v_y$, $u_y = -v_x$.

Solution: For (1):

$$\begin{aligned}\lim_{\Delta z \to 0} \left\{ \frac{1}{\Delta z} \left(\frac{1}{z+\Delta z} - \frac{1}{z} \right) \right\} &= \lim_{\Delta z \to 0} \left\{ \frac{1}{\Delta z} \frac{z - z - \Delta z}{(z+\Delta z)z} \right\} \\ &= \lim_{\Delta z \to 0} \left\{ \frac{-1}{(z+\Delta z)z} \right\} = -\frac{1}{z^2},\end{aligned}$$

proving that the limit exists independent of $\Delta z \to 0$.

For (2):

$$\frac{1}{z} = \frac{1}{z}\frac{\bar{z}}{\bar{z}} = \frac{x-iy}{x^2+y^2} = \frac{x}{r^2} - i\frac{y}{r^2},$$

so $u = \frac{x}{r^2}$ and $v = -\frac{y}{r^2}$. Thus

$$\begin{aligned} u_x &= \frac{1}{r^2} - \frac{2x}{r^3}\frac{x}{r} = \frac{1}{r^2} - \frac{2x^2}{r^4}; \quad u_y = -\frac{2xy}{r^4} \\ v_x &= \frac{2xy}{r^4}; \quad u_y = -\frac{1}{r^2} + \frac{2y^2}{r^4} \end{aligned}$$

Now clearly $u_y = -v_x$, and $u_x = v_y$ because

$$u_x - v_y = \frac{1}{r^2} - \frac{2x^2}{r^4} + \frac{1}{r^2} - \frac{2y^2}{r^4} = \frac{2}{r^2} - \frac{2(x^2+y^2)}{r^4} = \frac{2}{r^2} - \frac{2}{r^2} = 0,$$

so the Cauchy-Riemann equations hold.

Problem #4 (20pts): Let $f(z) = u(x,y) + iv(x,y)$ be a complex differentiable for all $z \in \mathcal{C}$, (so the Cauchy-Riemann equations hold), and let C be a curve that takes A to B.

(a) Derive, in terms of u and v, formulas for the real valued vector fields $\overrightarrow{G_1} = \overrightarrow{(M_1, N_1)}$ and $\overrightarrow{G_2} = \overrightarrow{(M_2, N_2)}$ such that

$$\int_C f(z)dz = \int_C \overrightarrow{G_1} \cdot \overrightarrow{T} ds + i \int_C \overrightarrow{G_2} \cdot \overrightarrow{T} ds,$$

where $\int_C \overrightarrow{G_i} \cdot \overrightarrow{T} ds$ are real line integrals on $\mathcal{R}^2$.

Solution:

$$\int_C f(z)dz = \int_C (u + iv)(dx + idy) = \int_C udx - vdy + i \int_C vdx + udy,$$

so $\overrightarrow{G_1} = \overrightarrow{(u, -v)}$ and $\overrightarrow{G_2} = \overrightarrow{(v, u)}$.

(b) Use the Cauchy-Riemann equations to prove that $\overrightarrow{G_1}$ and $\overrightarrow{G_2}$ are curl free, and state a theorem which implies that there exist $U(x,y)$ and $V(x,y)$ such that

$$\overrightarrow{G_1} = \nabla U, \quad \overrightarrow{G_2} = \nabla V.$$

Solution: $Curl(\overrightarrow{G}) = \mathbf{k}(N_x - M_y)$, so $\overrightarrow{G}$ is curl-free if $N_x - M_y = 0$. Now by C-R Eqns,

$$(N_1)_x - (M_1)_y = -v_x - u_y = 0; \quad (N_2)_x - (M_2)_y = u_x - v_y = 0,$$

so both $\overrightarrow{G_1}$ and $\overrightarrow{G_2}$ are curl-free.

Theorem (21D): If a vector field is curl-free in a simply connected domain, then the vector field is conservative. Thus we conclude that $\overrightarrow{G_1} = \nabla U$ and $\overrightarrow{G_2} = \nabla V$ for some scalar functions U, V.

(c) Letting $F(z) = U + iV$, prove

$$\int_C f(z)dz = F(B) - F(A).$$

(You may use any theorem from Mat21D which you can state correctly.)

Solution: Theorem (21D): If $\overrightarrow{G}$ is conservative with $\overrightarrow{G} = \nabla g$, then $\int_C \overrightarrow{G} \cdot \overrightarrow{T} ds = g(B) - g(A)$. Thus

$$\int_C f(z)dz = U(B) - U(A) + i(V(B) - V(A)).$$

Solution:
(d) Prove that $F(z)$ satisfies the Cauchy-Riemann equations, and $F'(z) = f(z)$.

Solution: We know

$$\nabla U = \overrightarrow{(U_x, U_y)} = \overrightarrow{(u, -v)}$$

and

$$\nabla V = \overrightarrow{(V_x, V_y)} = \overrightarrow{(v, u)},$$

so

$$U_x = u;\ U_y = -v;\ V_x = v;\ V_y = u$$

so $U_x = V_y$ and $U_y = -V_x$, implying that $F = U + iV$ satisfies the C-R equations. Finally, we know by the derivative being defined independent of $\Delta z \to 0$, taking $\Delta z = \Delta x$ gives $F'(z) = U_x + iV_x = u + iv = f(z)$ as claimed.

Problem #5 (20pts): Let C denote the simple closed curve given by the unit circle centered at $z = 0$, going counterclockwise around $z = 0$. Evaluate $\int_C \frac{dz}{z}$ directly by parameterization.

Solution: Let $x(t) = \cos t$ and $y(t) = \sin t$ so $z(t) = x(t) + iy(t) = \cos t + i \sin t$, $0 \le t \le 2\pi$ is a parameterization of C. Then...

$$\begin{aligned}
\int_C \frac{dz}{z} &= \int_C \frac{(x - iy)(dx + idy)}{x^2 + y^2} \\
&= \int_C \frac{xdx + ydy}{x^2 + y^2} + \int_C \frac{-ydx + xdy}{x^2 + y^2}.
\end{aligned}$$

On the unit circle, $x^2 + y^2 = 1$, $dx = -\sin t dt$, $dy = \cos t dt$, so

$$\begin{aligned}
\int_C \frac{dz}{z} &= \int_C -\cos t \sin t dt + \sin t \cos t dt + i \int_C \sin^2 t dt + \cos^2 t dt \\
&= 0 + i 2\pi = 2\pi i.
\end{aligned}$$