Name: ______________________________

Student ID#: ________________________

Section: ________________________

Midterm Exam 1

Wednesday, February 5

MAT 185A, Temple, Winter 2020

Print names and ID's clearly, and have your student ID ready to be checked when you turn in your exam. Write the solutions clearly and legibly. Do not write near the edge of the paper or the stapled corner. Show your work on every problem. Correct answers with no supporting work will not receive full credit. Be organized and use notation appropriately. No calculators, notes, books, cellphones, etc. may be used on this exam.

Problem	Your Score	Maximum Score
1		20
2		20
3		20
4		20
5		20
Total		100

Problem #1 (20pts): Recall that if $z = x + iy$, then $\bar{z} = x - iy$.
(a) Prove that $\overline{zw} = \bar{z}\bar{w}$.

Solution: Let $z = x + iy$, $w = u + iv$. Then

$$\bar{z}\bar{w} = (x - iy)(u - iv) = xu - yv - (yu + xv); \tag{1}$$

and

$$\overline{zw} = \overline{(x + iy)(u + iv)} = \overline{xu - yv + i(yu + xv)} = xu - yv - i(yu + xv). \tag{2}$$

(b) Recall $|z| = \sqrt{x^2 + y^2}$. Prove that $|zw| = |z||w|$.
Solution: Let $z = x + iy$, $w = u + iv$. Then

$$|zw|^2 = zw\overline{zw} = zw\bar{z}\bar{w} = z\bar{z}w\bar{w} = |z|^2|w|^2. \tag{3}$$

Taking square root gives the result.

Problem #2 (20pts): Let $\mathcal{C}$ denote a smooth simple closed counterclockwise curve which surrounds the point $z = i$. Argue in a picture that $\int_{\mathcal{C}} \frac{dz}{z-i} = \int_{\mathcal{C}_0} \frac{dz}{z-i}$ where $\mathcal{C}_0$ is the unit circle centered at i, and evaluate this latter integral by direct substitution.

Solution: $\int_{\mathcal{C}} \frac{dz}{z-i} = \int_{\mathcal{C}_0} \frac{dz}{z-i}$ where $\mathcal{C}_0$ is the counterclockwise unit circle centered at i. This follows because $\int_{\Gamma+\mathcal{C}_0-\Gamma+\mathcal{C}} \frac{dz}{z-i} = 0$ because $f(z) = \frac{1}{z-i}$ is analytic inside, as seen in picture. Now

$$\begin{aligned}
\int_{\mathcal{C}_0} \frac{dz}{z-i} &= \int_{\mathcal{C}_0} \frac{(dx+idy)}{(x+i(y-1))} = \int_{\mathcal{C}_0} \frac{(dx+idy)(x-i(y-1))}{x^2+(y-1)^2} \\
&= \int_{\mathcal{C}_0} \frac{xdx+(y-1)dy}{x^2+(y-1)^2} + i\int_{\mathcal{C}_0} \frac{-(y-1)dx+xdy}{x^2+(y-1)^2}. \qquad (4)
\end{aligned}$$

To evaluate, use parameterization $\mathbf{r}(t) = (x(t), y(t)) = (\cos t, 1+\sin t)$, $0 \le t \le 2\pi$, $dx = -\sin(t)dt$, $dy = \cos(t)dt$. Substituting into the above gives

$$\begin{aligned}
\int_{\mathcal{C}_0} \frac{dz}{z-i} &= \int_{\mathcal{C}_0} xdx + (y-1)dy + i\int_{\mathcal{C}_0} -(y-1)dx + xdy \\
&= \int_{\mathcal{C}_0} -\cos(t)\sin(t)dt + \sin(t)\cos(t)dt + i\int_{\mathcal{C}_0} \sin^2(t)dt + \cos^2(t)dt = 2\pi i.
\end{aligned}$$

Problem #3 (20pts): Let $w = f(z)$ with $w = u + iv$ and $z = x + iy$ where $f : \mathcal{C} \to \mathcal{C}$. Assume $f'(z)$ exists, by which we mean

$$\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = f'(z).$$

Derive the Cauchy Riemann equations from the condition that the limit is independent of $\Delta z \to 0$.

Solution: Taking $\Delta z = \Delta x$ gives

$$\lim_{\Delta x \to 0} \frac{f(z + \Delta x) - f(z)}{\Delta x} = \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x + \Delta x, y) - v(x, y))}{\Delta x} = u_x + iv_x.$$

Taking $\Delta z = \Delta y$ gives

$$\begin{aligned}
\lim_{i\Delta y \to 0} \frac{f(z + i\Delta y) - f(z)}{i\Delta y} &= \lim_{i\Delta y \to 0} \frac{u(x, y + i\Delta y) - u(x, y)}{i\Delta y} \left(\frac{i}{i}\right) + i \lim_{\Delta y \to 0} \frac{v(x, y + \Delta y) - v(x, y))}{i\Delta y} \\
&= \lim_{\Delta y \to 0} (-i) \frac{u(x, y + \Delta y) - u(x, y)}{\Delta y} + \lim_{\Delta y \to 0} \frac{v(x, y + \Delta y) - v(x, y))}{\Delta y} = v_y - iu_y.
\end{aligned}$$

Equating real and imaginary parts give the Cauchy Riemann equations: $u_x = v_y$, $v_x = -u_y$.

Problem #4 (20pts): Recall that because the exponential is 2π periodic in y, we had to restrict the range of the exponential to define the logarithm as its inverse. Without using any more than $e^{\log z} = z$ and $\frac{d}{dz}e^z = e^z$ together with general properties of derivatives,

Prove: $\frac{d}{dz} \log z = 1/z$. (State the property you use at each step of the proof.)

Solution: Assume $e^{\log z} = z$. Differentiating both sides with respect to z gives

$$\frac{d}{dz}e^{\log z} = \frac{d}{dz}z.$$

Using the Chain Rule on the LHS and the obvious $\frac{d}{dz}z = 1$ gives

$$e^{\log z}\frac{d}{dz}\log z = 1.$$

Solving for $\frac{d}{dz}\log z$ gives

$$\frac{d}{dz}\log z = \frac{1}{e^{\log z}} = \frac{1}{z}.$$

Problem #5 (20pts): Assume $F(z) = U + iV$ and $f(z) = u + iv$ are entire functions such that $F'(z) = f(z)$.

(a) Derive the line integrals for the real and imaginary parts of $\int_{\mathcal{C}} f(z)dz$.

Solution:

$$\int_{\mathcal{C}} f(z)dz = \int_{\mathcal{C}} (u+iv)(dx+idy) = \int_{\mathcal{C}} udx - vdy + i\int_{\mathcal{C}} vdx + udy$$

(b) Using only the definition of derivative, Cauchy Riemann equations and properties of line integrals from Mat 21D, prove that the Fundamental Theorem of Calculus holds:

$$\int_{\mathcal{C}} f(z)dz = F(B) - F(A)$$

where C is any smooth curve in the xy-plane taking A to B.

Solution:

$$\int_{\mathcal{C}} f(z)dz = \int_{\mathcal{C}} \mathbf{G}_1 \cdot \mathbf{T}ds + i\int_{\mathcal{C}} \mathbf{G}_2 \cdot \mathbf{T}ds$$

with $\mathbf{G}_1 = \overrightarrow{(u,-v)}$, $\mathbf{G}_2 = \overrightarrow{(v,u)}$. Now since $F' = f$, taking $\Delta z = \Delta x$ gives $U_x = u$, $V_x = v$. The Cauchy Riemann equations for F then give $U_x = V_y = u$, $V_x = -U_y = v$. Thus

$$\nabla U = \overrightarrow{(U_x, U_y)} = \overrightarrow{(u,-v)} = \mathbf{G_1},\ \nabla V = \overrightarrow{(V_x, V_y)} = \overrightarrow{(v,u)} = \mathbf{G_2}.$$

Thus

$$\begin{aligned}
\int_{\mathcal{C}} f(z)dz &= \int_{\mathcal{C}} \nabla U \cdot \mathbf{T}ds + i\int_{\mathcal{C}} \nabla V \cdot \mathbf{T}ds \\
&= \int_{t_A}^{t_B} \frac{d}{dt}U(\mathbf{r}(t))dt + i\int_{t_A}^{t_B} \frac{d}{dt}V(\mathbf{r}(t))dt \\
&= U(B) - U(A) + iV(B) - iV(A) = F(B) - F(A),
\end{aligned}$$

where $\mathbf{r}(t)$ is a parameterization of $\mathcal{C}$ between $\mathbf{r}(t_A) = A$ and $\mathbf{r}(t_B) = B$ for $t_A \leq t \leq t_B$.