Name: ______________________________

Student ID#: ________________________

Section: ________________________

Midterm Exam 2

Wednesday, March 6

MAT 185A, Temple, Winter 2019

Print names and ID's clearly, and have your student ID ready to be checked when you turn in your exam. Write the solutions clearly and legibly. Do not write near the edge of the paper or the stapled corner. Show your work on every problem. Correct answers with no supporting work will not receive full credit. Be organized and use notation appropriately. No calculators, notes, books, cellphones, etc. may be used on this exam.

Problem	Your Score	Maximum Score
1		20
2		20
3		20
4		20
5		20
Total		100

Problem #1 (20pts): Recall that $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$, for $z = x + iy \in \mathcal{C}$.

(a) Show that $\sin z$ reduces to $\sin x$ when $z = x \in \mathcal{R}$.

Solution: $e^{ix} = \cos x + i \sin x$, so

$$\frac{e^{ix} - e^{-ix}}{2i} = \frac{\cos x + i \sin x - \cos(-x) - i \sin(-x)}{2i} = \frac{2i \sin x}{2i} = \sin x$$

(b) Find $u(x, y)$ and $v(x, y)$ real so that $\sin(z) = u(x, y) + iv(x, y)$.

Solution:

$$\begin{aligned} \sin(x+iy) &= \frac{e^{ix-y} - e^{-ix+y}}{2i} = \frac{e^{-y}(\cos x + i \sin x) - e^{y}(\cos(-x) + i \sin(-x))}{2i} \\ &= \frac{e^{y} + e^{-y}}{2} \sin x + i \frac{e^{y} - e^{-y}}{2} \cos x \end{aligned}$$

so

$$u = \cosh(y) \sin(x), \quad v = \sinh(y) \cos(x)$$

(c) Prove $f(z) = \sin z$ satisfies the Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$.

Solution:

$$\begin{aligned} u_x &= \cosh(y) \cos(x), \quad v_y = \cosh(y) \cos(x) \\ u_y &= \sinh(y) \cos(x), \quad v_x = -\sinh(y) \cos(x) \end{aligned}$$

Problem #2 (20pts):

(a) Assume that f^{-1} and f are inverses of each other, and $w = f^{-1}(z)$. Prove that

$$\frac{d}{dz} f^{-1}(z) = \frac{1}{\frac{d}{dw} f(w)}. \quad [\textit{Corrected on BB during exam}]$$

Solution: Since $f(f^{-1}(z)) = z$, differentiating both sides and using the Chain Rule gives

$$f'(f^{-1}(z)) \frac{d}{dz} f^{-1}(z) = 1.$$

Thus $\frac{d}{dz} f^{-1}(z) = 1/f'(f^{-1}(z)) = 1/f'(w)$.

(b) The *logarithm* is defined as the inverse of the exponential, so $w = log(z)$ if and only if $z = e^w$. Use part **(a)** together with properties of the exponential to derive $\frac{d}{dz} \log z$.

Solution: $\frac{d}{dz} \log z = \frac{1}{\frac{d}{dw} e^w} = \frac{1}{e^w} = 1/z$.

Problem #3 (20pts): Let $z = 2i$, and let $n \geq 1$ be an integer. Find $z^{1/n}$. (That is, find all complex numbers w such that $w^n = z$.) How many of them can be real numbers? Explain.

Solution:

$$z^{1/n} = e^{\frac{1}{n}\log(z)} = e^{\frac{1}{n}\log(r)+i\frac{\pi/2+2k\pi}{n}} = r^{1/n}e^{i\frac{\pi/2+2k\pi}{n}}$$

for $k = 0, ..., n-1$. The n angles are

$$\theta_k = \frac{\pi/2 + 2k\pi}{n} = \frac{1+4k}{n}\frac{\pi}{2}.$$

Only two of the roots can be real, corresponding to the possibilities $\theta_k = 0, \pi$.

Problem #4 (20pts): Assume $f(z) = u + iv$ is analytic in an open set containing the closure of the ball $B_R(z_0)$, and let C_R denote the positively oriented closed curve which is its boundary.

(a) Prove that v at the center is given by its average value, i.e., prove

$$v(z_0) = \frac{1}{2\pi}\int_0^{2\pi} v(z_0 + Re^{it})dt.$$

Solution: By the Cauchy integral formula,

$$f(z_0) = \frac{1}{2\pi i}\int_{C_R} \frac{f(w)}{w - z_0}dw.$$

Using $z(t) = z_0 + Re^t$, $0 \leq t \leq 2\pi$, $dz = iRe^{it}dt$ we have

$$f(z_0) = \frac{1}{2\pi i}\int_{C_R} \frac{f(z_0 + Re^{it})}{w - z_0}dw = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + Re^{it})dt.$$

Taking the imaginary part gives the desired answer.

(b) Now assume only that f is continuous, but that for any points A, B in the complex plane, $\int_C f(z)dz$ is independent of path C taking A to B. Let point A be fixed. Prove: $F(z) = \int_A^z f(z)dz$ is an anti-derivative of f. (Here $\int_A^z$ denotes the integral along any path from A to z.)

Solution: Since integration can be along any path,

$$\begin{aligned}
\left|\frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z)\right| &= \left|\frac{1}{\Delta z}\left(\int_z^{z+\Delta z} f(w)dw\right) - f(z) frac1\Delta z \int_z^{z+\Delta z} dw\right| \\
&= \left|\frac{1}{\Delta z}\left(\int_z^{z+\Delta z} f(w) - f(z)dw\right)\right| \\
&\leq \frac{1}{|\Delta z|}\int_z^{z+\Delta z} |f(w)-f(z)||dw| \\
&\leq \frac{1}{|\Delta z|}|\Delta z| Max_{|w-z|\leq|\Delta z|}|f(w)-f(z)| \to 0 \quad as \quad \Delta z \to 0
\end{aligned}$$

because f is continuous at z. Thus $F'(z) = f(z)$ by definition.

Problem #5 (20pts): Recall that Cauchy's Inequality states that if f is analytic in a neighborhood of $\overline{B_R(z_0)}$, then $|f^{(k)}(z_0)| \leq \frac{k!}{R^k}M$, where M is the maximum value of f in $\overline{B_R(z_0)}$. (Here $B_R(z_0)$ denotes the open ball with center z_0 and radius R, and the bar on top denotes its closure.)

(a) Use Cauchy's Inequality to prove Liouville's Theorem, that every bounded entire function is constant.

Solution: Assume $f(z)$ is bounded and entire. Bounded means there exists $M > 0$ such that $f(z) \leq M$ for all $z \in \mathcal{C}$. Entire means we can apply Cauchy's Inequality to $B_R(z)$ for any $R > 0$, any z. In the case $k = 1$ this is

$$|f'(z)| \leq \frac{1}{R^k}M \to 0 \quad as \quad R \to \infty.$$

Thus $f'(z) = 0$ for all z, implying $f(z) = const.$

(b) Use Liouville's Theorem to prove that every polynomial $P(z)$ of order $n \geq 1$ has a complex root. (You may assume that every polynomial $P(z)$ is non-constant and $lim_{z\to\infty} P(z) = \infty$ when $n \geq 1$.)

Solution: Assume $P(z)$ is a polynomial and $P(z) \neq 0$ for all $z \in \mathcal{C}$. Then $1/P(z)$ is analytic. Since $lim_{z\to\infty} P(z) = \infty$, there exists an $R > 0$ such that $P(z) \geq 1$ for $|z| > R$. Thus $|1/P(z)| \leq 1$ for all $|z| \geq R$ But $1/P(z)$ is continuous and hence bounded on the closed ball $|z| \leq R$, so there exists M' such that $|P(z)| \leq M'$ for $|z| \leq R$. It follows that $|1/P(z)| \leq 1 + M'$, so $1/P(z)$ is a bounded entire function, and hence constant by Liouville's Theorem. Conclude that if $P(z)$ is not constant, then $P(z)$ must have a root.