Midterm II Solns - MAT 185A W2020

Problem #1 (20pts): Recall that $\cos z = \frac{e^{iz}+e^{-iz}}{2}$, for $z = x + iy \in \mathcal{C}$.

(a) Show that $\cos z$ reduces to $\cos x$ when $z = x \in \mathcal{R}$.

Solution: $e^{ix} = \cos x + i\sin x$, so

$$\frac{e^{ix}+e^{-ix}}{2} = \frac{\cos x + i\sin x + \cos(-x) + i\sin(-x)}{2} = \frac{2\cos x}{2} = \cos x$$

(b) Find $u(x,y)$ and $v(x,y)$ real so that $\cos(z) = u(x,y) + iv(x,y)$.

Solution:

$$\begin{aligned} \cos(x+iy) &= \frac{e^{ix-y}+e^{-ix+y}}{2} = \frac{e^{-y}(\cos x + i\sin x) + e^{y}(\cos(-x) + i\sin(-x))}{2} \\ &= \frac{e^{y}+e^{-y}}{2}\cos x - \frac{e^{y}-e^{-y}}{i2}\sin x \end{aligned}$$

so

$$u = \cosh(y)\cos(x), \quad v = -\sinh(y)\sin(x)$$

(c) Prove $f(z) = \cos z$ satisfies the Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$.

Solution:

$$\begin{aligned} u_x &= -\cosh(y)\sin(x), \quad v_y = -\cosh(y)\sin(x) \\ u_y &= \sinh(y)\cos(x), \quad v_x = -\sinh(y)\cos(x) \end{aligned}$$

#2 Solution:

$$i^{1/\pi} = e^{\frac{1}{\pi}\log(i)} = e^{\frac{1}{\pi} i(\pi/2 + 2k\pi)} = e^{i(\frac{1}{2}+2k)}$$

Claim - These are all distinct roots.

I.e., if not, then $\frac{1}{2}+2k_1 = \frac{1}{2}+2k_2+2m\pi$ for some integers k_1, k_2, m. But then

$$\frac{k_1-k_2}{m} = \pi \Rightarrow \pi \text{ rational. } \#$$

#3 Solution:

(a) $$f'(z) = \frac{1}{2\pi i} \int_{C_R} \frac{f(w)}{(w-z)^2}\, dw$$

$$|f'(z)| \le \left|\frac{1}{2\pi}\right| |C_R| \max_{C_R} \left|\frac{f(w)}{(w-z)^2}\right|$$

$M = \max_{C_R} |f|$

$$\le \left|\frac{1}{2\pi}\right| 2\pi R \frac{M}{R^2} = \frac{M}{R} \checkmark$$

(b) Assume $|f(z)| \le M$. Then

$|f'(z)| \le \frac{M}{R}$ holds $\forall\, R>0$, every $z \in \mathbb{C}$

$\therefore\ |f'(z)| = 0 \Rightarrow f'(z) = 0 \Rightarrow f(z) = \text{const}.$

#4 Solution:

(a) CR

$$u_x = v_y \qquad u_{xx} = v_{xy} \qquad v_{xx} = -u_{yx}$$

$$v_x = -u_y \qquad \underline{u_{yy} = -v_{xy}} \qquad \underline{v_{yy} = u_{xy}}$$

$$u_{xx} + u_{yy} = 0 \qquad v_{xx} + v_{yy} = 0$$

(b)

$$f(0) = \frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{w}\, dw \qquad w = e^{it}, \quad dw = ie^{it}\,dt, \quad 0 \leq t < 2\pi$$

$$= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(e^{it})}{e^{it}}\, ie^{it}\,dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(e^{it})\,dt$$

$$u(0) + iv(0) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{it})\,dt + \frac{i}{2\pi} \int_0^{2\pi} v(e^{it})\,dt$$

$$\Rightarrow u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{it})\,dt \quad \checkmark$$

#5 Solution:

(a) $P(z) = a_n z^n + \cdots + a_0$

$$|P(z)| \geq |a_n z^n| - |a_{n-1} z^{n-1} + \cdots + a_0|$$

Letting $z = re^{i\theta}$ gives (wlog $r \geq 1$)

$$|P(z)| \geq |a_n| r^n - \left(|a_{n-1}| r^{n-1} + \cdots + |a_0|\right.$$

$$\geq 2$$

if $$r > \frac{|a_{n-1}| r^{n-1} + \cdots + |a_0|}{|a_n| r^{n-1}} + 2$$

so $$r > \frac{|a_{n-1}| + \cdots + |a_0|}{|a_n|} + 2 > \frac{|a_{n-1}| r^{n-1} + \cdots + |a_0|}{|a_n| r^{n-1}} + 2$$

$\uparrow$

suffices

(#5) (b) Assume $P(z) \neq 0$. Then

$$f(z) = \frac{1}{|P(z)|} \text{ is entire.}$$

But by (a) $\exists R$ st $|z| \geq R$ implies

$$|P(z)| \leq 2$$

So

$$\frac{1}{|P(z)|} \leq \frac{1}{2}.$$

$|P(z)|^{-1}$ cont $\Rightarrow$ $|P(z)|^{-1}$ takes a max value M on $\overline{B_R(0)}$ compact. Thus

$$\frac{1}{|P(z)|} \leq \text{Max}\{\tfrac{1}{2}, M\}.$$

$\Rightarrow \frac{1}{|P(z)|}$ bounded entire $\Rightarrow$ const. $\#$.