# 10A– Three Generalizations of the Fundamental Theorem of Calculus

## MATH 22C

### 1. Introduction

In the next four sections we present applications of the three generalizations of the Fundamental Theorem of Calculus (FTC) to three space dimensions $(x, y, z) \in \mathcal{R}^3$, a version associated with each of the three linear operators, the *Gradient*, the *Curl* and the *Divergence*. Since much of classical physics is framed in terms of these three generalizations of FTC, these operators are often referred to as the three linear first order operators of classical physics.

The FTC in one dimension states that the integral of a function over a closed interval $[a, b]$ is equal to its anti-derivative evaluated between the endpoints of the interval:

$$\int_a^b f'(x)dx = f(b) - f(a).$$

This generalizes to the following three versions of the FTC in two and three dimensions. The first states that the line integral of a gradient vector field $\mathbf{F} = \nabla f$ along a curve $\mathcal{C}$, (in physics the *work done* by $\mathbf{F}$) is exactly equal to the change in its potential potential $f$ across the endpoints $A$, $B$ of $\mathcal{C}$:

$$\int_{\mathcal{C}} \mathbf{F} \cdot T\, ds = f(B) - f(A). \tag{1}$$

The second, called Stokes Theorem, says that the flux of the Curl of a vector field $F$ through a two dimensional surface $\mathcal{S}$ in $\mathcal{R}^3$ is the line integral of $F$ around the curve $\mathcal{C}$ that bounds $\mathcal{S}$:

$$\int\int_{\mathcal{S}} Curl\mathbf{F}\cdot\mathbf{n}\,d\sigma = \int_{\mathcal{C}} \mathbf{F}\cdot T\,ds \tag{2}$$

And the third, called the Divergence Theorem, states that the integral of the Divergence of $\mathbf{F}$ over an enclosed volume $\mathcal{V}$ is equal to the flux of $F$ outward through the two dimensional closed surface $\mathcal{S}$ that bounds $\mathcal{V}$:

$$\int\int\int_{\mathcal{V}} Div\mathbf{F}\,dV = \int\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{n}\,d\sigma. \tag{3}$$

## 2. The three linear first order operators of classical physics

The three linear partial differential operators of classical physics are the Gradient=$\nabla$, the Curl=$\nabla\times$ and the Divergence=$\nabla\cdot$. That is, formally defining

$$\nabla = \left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right) \equiv (\partial_x,\partial_y,\partial_z)\,,$$

the Gradient of a scalar function $f(x,y,z)\equiv f(\mathbf{x})$ becomes

$$\nabla f = \left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right);$$

and for a vector field

$$\mathbf{F}(\mathbf{x}) = (M(\mathbf{x}),N(\mathbf{x}),P(\mathbf{x})),$$

the Curl and Divergence are defined by

$$\begin{aligned} Curl(\mathbf{F}) &= \nabla\times\mathbf{F} = Det\begin{vmatrix} \mathbf{i} & \mathbf{j} & ,\mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix} \qquad (4)\\ &= \mathbf{i}(P_y - N_z) - \mathbf{j}(P_y - M_z) + \mathbf{k}(N_x - M_y), \end{aligned}$$

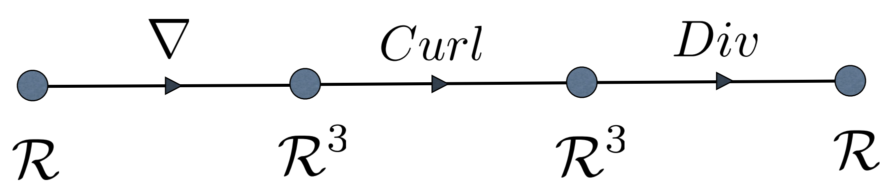
$$Div(\mathbf{F}) = \nabla\cdot\mathbf{F} = M_x + N_y + P_z. \tag{5}$$

Note that the $Curl(\mathbf{F})$ is a vector field and the $Div(\mathbf{F})$ is a scalar function.

The three first order differential operators $\nabla$, $Curl$, $Div$ of classical physics are related in a remarkable way as diagrammed in Figure A. This is a snapshot way of seeing that $\nabla$ takes scalar functions with values in $\mathcal{R}$ to vector valued functions with values in $\mathcal{R}^3$; $Curl$ takes vector valued functions with values in $\mathcal{R}^3$ to vector valued functions with values in $\mathcal{R}^3$; and $Div$ takes vector valued functions with values in $\mathcal{R}^3$ to scalar valued functions with values in $\mathcal{R}$. The diagram indicates that when written in this order, taking any two in a row makes zero. This is really two identities:

$$Curl(\nabla f) = 0 \tag{6}$$

$$Div(Curl\mathbf{F}) = 0. \tag{7}$$




**Three first order linear differential operators of Classical Physics:**

**(1) Two in a row make zero.**

**(2) Only Curls solve Div=0, and only Gradients solve Curl=0.**

**Figure** A

Moreover, an important theorem of vector calculus states a converse of this. Namely, if a vector field "has no singularites" (i.e., is defined and smooth everywhere in a domain with no holes), then: (i) If $Curl\mathbf{F} = 0$ then $\mathbf{F} = \nabla f$ for some scalar $f$; and (ii) If $Div\mathbf{F}=0$, then $\mathbf{F}=Curl\mathbf{G}$ for some vector valued function $\mathbf{G}$.

In this section we give an application of the first version of FTC in (1) together with (i) to explain why integrals of complex valued functions make sense as line integrals independent of path. We start by introducing a fundamental vector field $\mathbf{G}$ that is Curl free, but has a singularity along the $z$-axis. We see that the presence of this singularity creates a non-zero contribution to line integrals on curves that encircles the $z$-axis, but that this contribution is the same for every such path. We conclude by showing that this effect explains the essence of the Residue Theorem of Complex Variables. In the next section we study Stokes Theorem and the Divergence Theorem in the context of Maxwell's equations of electromagnetism.

## 3. Line integrals and conservative vector fields

We start by recalling the four equivalent version of the line integral of $\mathbf{F}=M\mathbf{i}+N\mathbf{j}+P\mathbf{k}$, where $M,N,P$ are general functions of $(x,y,z)$ giving the components of $\mathbf{F}$ at each point:

$$\begin{aligned}\int_{\mathcal{C}}\mathbf{F}\cdot\mathbf{T}\,ds &= \int_{\mathcal{C}}\mathbf{F}\cdot\mathbf{dr}=\int_{\mathcal{C}}Mdx+Ndy+Pdz\\ &= \int_a^b\mathbf{F}\cdot\mathbf{v}\,dt.\end{aligned}\tag{8}$$

These equivalences follow easily from the Leibniz notation associated with a parameterization $\mathbf{r}(t)=x(t)\mathbf{i}+y(t)\mathbf{j}+z(t)\mathbf{k}$ of the curve $\mathcal{C}$, $a\le t\le b$ by using $\frac{d\mathbf{r}}{dt}=\mathbf{v}$ and $\frac{ds}{dt}=\|\mathbf{v}\|$ and writing:

$$\mathbf{T}ds=\mathbf{v}dt;\quad d\mathbf{r}=\mathbf{v}dt=dx\mathbf{i}+dy\mathbf{j}+dz\mathbf{k}.$$

Note that the final form tells how to use a parameterization of the $\mathcal{C}$ (i.e., a coordinate system on the curve), to compute the value of the line integral. A vector field $\mathbf{F}$ is called *conservative* if there exists a scalar function $f$ such that $\mathbf{F} = \nabla f$. Most vector fields are not conservative, but when they are, we have a conservation of energy principle. Indeed, by the chain rule, we know

$$\frac{d}{dt} f(\mathbf{r}(t)) = \nabla f \cdot \mathbf{v}(t),$$

so using this in the fourth version of the line integral in (8) yields

$$\begin{aligned} \int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T}\, ds &= \int_a^b \mathbf{F} \cdot \mathbf{v}\, dt = \int_a^b \frac{d}{dt} f(\mathbf{r}(t))\, dt \qquad (9) \\ &= f(\mathbf{r}(b)) - f(\mathbf{r}(a)) = f(B) - f(A) \end{aligned}$$

which is our first version (1) of FTC.

In physics, if $\mathbf{F} = \nabla f$ is a forcefield, then we call it a conservative force field, and we define $P = -f$ as the potential energy. Then (9) says the work done by a conservative forcefield is minus the change in potential energy. Now according to Newton's laws of motion, many forces can be acting during the motion an object of mass $m$, but the total force that gives the acceleration $\mathbf{a} = \frac{d\mathbf{v}}{dt}$ is the sum of all the separate forces. If the total force is conservative, say $\mathbf{F} = \nabla f$ where $\mathbf{F}$ is the total force, then we can derive a second expression for the work done as the change in kinetic energy, namely,

$$\begin{aligned} \int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T}\, ds &= \int_a^b \mathbf{F} \cdot \mathbf{v}\, dt = \int_a^b m \frac{d\mathbf{v}}{dt} \cdot \mathbf{v}\, dt \qquad (10) \\ &= \frac{m}{2} \int_a^b \frac{d}{dt} (\mathbf{v} \cdot \mathbf{v})\, dt = \frac{1}{2} m \|\mathbf{v}_B\|^2 - \frac{1}{2} m \|\mathbf{v}_A\|^2. \end{aligned}$$

Putting (9) and (10) together, we get the principle that the energy, the sum of kinetic plus potential, is conserved for motion $\mathbf{r}(t)$ in a conservative force field. I.e., defining the energy,

$$E(t) = \frac{1}{2}m\|\mathbf{v}(t)\|^2 + P(\mathbf{r}(t)),$$

we have that $E(t) = \textit{constant}$ all along the motion $\mathbf{r}(t)$.

A great example of a conservative forcefield is the central force exerted by the sun on the planets under the assumption of Newton's inverse square force law:

$$\mathbf{F} = -GM_SM_P\frac{1}{r^2}\left(\frac{\mathbf{r}}{r}\right) = -\mathcal{G}\frac{\mathbf{r}}{r^3}. \tag{11}$$

Here $\mathbf{F}$ is the force exerted by the sun on the plane $P$, $\mathcal{G} = GM_SM_P$, the product of Newton's gravitational constant, the mass of the sun, and the mass of the planet, respectively, is assumed constant, $r = \sqrt{x^2 + y^2 + z^2}$ is the function of $(x, y, z)$ that gives the length of the position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, and we take the sun as a fixed center of the rectilinear coordinate system. Now functions that exhibit spherical symmetry depend on $r$, and it is useful to remember the easily derivable formula

$$\frac{\partial r}{\partial x} = \frac{x}{r}; \quad \frac{\partial r}{\partial y} = \frac{y}{r}; \quad \frac{\partial r}{\partial z} = \frac{z}{r}. \tag{12}$$

(More generally, if $r = \sqrt{\sum_{i=1}^n x_i^2}$, then $\partial r/\partial x_i = x_i/r$.) Using (12) we can easily compute the following gradient:

$$\nabla\left(\frac{\mathcal{G}}{r}\right) = -\mathcal{G}\left(\frac{x}{r^3}\mathbf{i} + \frac{y}{r^3}\mathbf{j} + \frac{z}{r^3}\mathbf{k}\right) = \mathbf{F}.$$

This demonstrates that (11) is a conservative forcefield, and conservation of energy holds along all motions.

We next recall the basic theorem of vector calculus regarding conservative vector fields:

**Theorem 1.** *Let* $\mathbf{F}$ *be a smooth vector field defined on* $\mathcal{R}^3$*. Then the following are equivalent:*

**(i)** *There exists a scalar function* $f$ *such that* $\mathbf{F} = \nabla f$*.*

**(ii)** $\int_{\mathcal{C}} \mathbf{F}\cdot\mathbf{T}\,ds = 0$ *for every simple closed curve* $\mathcal{C}$*.*

In particular, if (ii) holds, then line integrals are independent of path, and it is not difficult to show that, fixing point $A$, defining $f$ to be the integral to any other point $B$, determines the $f$ such that $\mathbf{F} = \nabla f$ to within a constant. Stokes theorem then gives us a condition under which $(ii)$ and hence $(i)$ holds, and hence gives us a condition under which a vector field is conservative. Indeed, if $\mathcal{C}$ is a closed curve, and $Curl\mathbf{F} = 0$ on a surface $\mathcal{S}$ which has $\mathcal{C}$ as its boundary, then by Stokes theorem (2) we have

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T}\, ds = \int \int_{\mathcal{S}} Curl\mathbf{F}\, d\sigma = 0,$$

verifying $(ii)$ and hence $(i)$. In particular, a *simply connected* domain of $\mathcal{R}^3$ is defined as a region in which *every simple closed curve passing through a point can be continuously deformed to the point without passing out of the domain.* Thus this is just a condition for every simple closed curve in the domain to have a surface with that curve as its boundary, such that the surface lies entirely within the domain–meaning that Stokes Theorem can be applied to that surface. From this follows the third characterization of conservative vector fields.

**Theorem 2.** *Let* $\mathbf{F}$ *be a smooth vector field such that* $Curl\mathbf{F} = 0$ *in a simply connected domain of* $\mathcal{R}^3$*. Then there exists a scalar function* $f$ *such that* $\mathbf{F} = \nabla f$*.*

We now consider a fundamental vector field which is Curl-free, but *not* in a simply connected domain due to the presence of a *singularity* on the $z$-axis. The following vector field is defined in the $xy$-plane, but we view as a vector field in $\mathbf{R}^3$ with a zero $z$-component:

$$\mathbf{H} = \frac{-y}{x^2 + y^2}\mathbf{i} + \frac{x}{x^2 + y^2}\mathbf{j} + 0\mathbf{k}. \tag{13}$$

Since the denominators contain $r = \sqrt{x^2 + y^2}$ which tends to zero as $x$ and $y$ tends to zero, it follows that $\mathbf{H}$ is undefined, or has a singularity at $r = 0$, which is the $z$-axis. Using (12), we can easily use (14) to take the Curl($\mathbf{H}$):

$$\begin{aligned} Curl(\mathbf{H}) &= \nabla \times \mathbf{H} = Det \begin{vmatrix} \mathbf{i} & \mathbf{j} & ,\mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ \frac{-y}{r^2} & \frac{x}{r^2} & 0 \end{vmatrix} \qquad (14) \\ &= \mathbf{i}(0) - \mathbf{j}(0) + \mathbf{k}\left\{\left(\frac{1}{r^2} - \frac{2x^2}{r^4}\right) + \left(\frac{1}{r^2} - \frac{2y^2}{r^4}\right)\right\} \\ &= \mathbf{k}\left\{\left(\frac{2}{r^2} - \frac{2x^2}{r^4} - \frac{2y^2}{r^4}\right)\right\} = \mathbf{k}\left(\frac{2}{r^2} - \frac{2}{r^2}\right) = 0. \end{aligned}$$

Now if the $Curl(\mathbf{H}) = 0$ in a region where Stokes Theorem holds, we can again prove that the integral around any closed curve $\mathcal{C}$ is zero by taking any two dimensional surface $\mathcal{S}$ which is enclosed by $\mathcal{C}$, and deduce

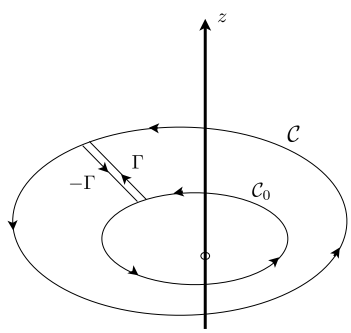
$$\int_{\mathcal{C}} \mathbf{H} \cdot \mathbf{T}\, ds = \int \int_{\mathcal{S}} Curl\mathbf{H} \cdot \mathbf{n}\, d\sigma = 0.$$

But since our $\mathbf{H}$ is singular on the $z$-axis (because $r$ is then zero in the denominator), we cannot apply this when the curve $\mathbf{C}$ encirles the $z$-axis. This is the explanation for the condition that we must have $Curl\mathbf{H} = 0$ in a *simply connected domain* to imply $\mathbf{H} = \nabla f$, i.e., that is exactly the condition for there to exist a surface $\mathcal{S}$ with boundary $\mathcal{C}$ on which $Curl\mathbf{H} = 0$. We now prove that the line integral of $\mathbf{H}$ is *the same* around any curve that encircles the $z$-axis once around, counterclockwise, and in fact, its value is $2\pi$. Lets first see that all such curves must produce the same line integral. For this, given one simple closed curve $\mathcal{C}_0$ that encircles the $z$-axis counterclockwise exactly once, and any other such curve $\mathcal{C}$, consider the concatenation of curves $\bar{\mathcal{C}} = \mathcal{C}_0 \cup \Gamma \cup -\mathcal{C} \cup -\Gamma$, diagrammed in Figure 1. Then $\bar{\mathcal{C}}$ does not encircle the $z$-axis, so there is a surface on which $Curl\mathbf{H} = 0$ having $\bar{\mathcal{C}}$ as its boundary, and so Stokes Theorem gives us

$$\begin{aligned} 0 &= \int_{\bar{\mathcal{C}}} \mathbf{H}\cdot\mathbf{T}\,ds = \int_{\mathcal{C}_0\cup\Gamma\cup-\mathcal{C}\cup-\Gamma} \mathbf{H}\cdot\mathbf{T}\,ds \qquad (15)\\ &= \int_{\mathcal{C}_0} + \int_{\Gamma} - \int_{\mathcal{C}} - \int_{\Gamma}\\ &= \int_{\mathcal{C}_0} - \int_{\mathcal{C}} \end{aligned}$$

and hence we conclude

$$\int_{\mathcal{C}} \mathbf{H}\cdot\mathbf{T}\,ds = \int_{\mathcal{C}_0} \mathbf{H}\cdot\mathbf{T}\,ds. \qquad (16)$$




$\bar{\mathcal{C}} = \mathcal{C}_0 \cup \Gamma \cup -\mathcal{C} \cup -\Gamma$

**Figure** 1

It remains, then, only to evaluate the line integral on one specific such curve $\mathcal{C}_0$ simple enough to do the evaluation. Taking the circle of center $x = y = 0$ radius $r = 1$ and $z = 0$ we can use the parameterization $\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j}$ to compute

$$\int_{\mathcal{C}_0} \mathbf{H}\cdot\mathbf{T}\,ds = \int_0^{2\pi} \overrightarrow{(-\sin(t),\cos(t))}\cdot\overrightarrow{(-\sin(t),\cos(t))} = 2\pi.$$

We conclude that for every simple closed curve $\mathcal{C}$ encircling the $z$-axis once counterclockwise, the line integral of (13) is given by

$$\int_{\mathcal{C}} \mathbf{H} \cdot \mathbf{T}\, ds = 2\pi. \tag{17}$$

Interestingly, not every Curl-free vector field with a singularity on the $z$-axis will have a nonzero integral on paths that encircle the $z$-axis. An example is the vector field

$$\mathbf{G} = \frac{x}{r^2}\mathbf{i} + \frac{y}{r^2}\mathbf{j}, \tag{18}$$

where on every closed curve, including those that encircle the origin,

$$\int_{\mathcal{C}} \mathbf{G} \cdot \mathbf{T}\, ds = 0. \tag{19}$$

The reader can verify this on the unit circle using the same parameterization as above, and then use the same argument to show that the answer is the same for every path that circles the origin.

## 4. Application to complex analytic functions

To finish, we apply our results to see the following miracle of mathematics: that the Cauchy-Riemann equations, the equations which guarantee a complex valued function has a well defined derivative, are also exactly the Curl-free conditions required to make the line integral of the complex valued function independent of path. We then use the vector fields $G$ and $H$ in (18), (17) to explain the *Residue Theorem* of complex variables. The point we wish to make is that this is the *entry point* to Complex Variables. It shows in one glimpse, by the simplest route, what the essential miracle is, and how one might discover it. After passing through this doorway, all of the results of complex variables are on the other side, and really, after this, what could be the surprise?

Writing a complex number as $z = x + iy$ (not to be confused with the $z$-axis!) is simply a way of writing an ordered pair $(x, y)$ in $\mathcal{R}^2$ so as to give it the same scalar multiplication and additive structure, but the use of $i$ augments this by defining a way to multiply vectors using the distributive property of multiplication over addition together with the defining property $i^2 = -1$. A complex valued function is then given by

$$f(z) = u(x, y) + iv(x, y),$$

the function being determined by the two real valued functions $u(x, y)$ and $v(x, y)$ which give it real and imaginary parts, respectively. We say that the function is analytic if the derivative

$$\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}. \tag{20}$$

exists for any way of taking the limit $\Delta z = \Delta x + i\Delta y \to 0$. If we choose $\Delta z = \Delta x$ then (20) implies

$$\begin{aligned} &\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \qquad (21)\\ &= \lim_{\Delta x \to 0} \left\{ \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + i \frac{v(x + \Delta x, y) - v(x, y)}{\Delta x} \right\} \\ &\quad = u_x + iv_x. \end{aligned}$$

On the other hand, if we pick $\Delta z = i\Delta y$, then then (20) implies

$$\begin{aligned} &\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \qquad (22) \\ &= \lim_{\Delta x \to 0} \left\{ \frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + i\frac{v(x,y) - v(x,y+\Delta y)}{i\Delta y} \right\} \\ &= \lim_{\Delta x \to 0} \left\{ \frac{v(x,y) - v(x,y+\Delta y)}{\Delta y} - i\frac{u(x,y+\Delta y) - u(x,y)}{\Delta y} \right\} \\ &\quad = v_y - iu_y. \end{aligned}$$

For the limit in (21) to agree with the limit in (22), $u$ and $v$ must meet the conditions

$$u_x = v_y, \quad v_x = -u_y. \qquad (23)$$

Equations (23) are called the Cauchy-Riemann equations, and it turns out that this necessary condition is sufficient also to guarantee the function $f$ has a well defined complex derivative, and in fact, derivatives of all orders. Our purpose here is to see that (23) are exactly the Curl-free conditions required to make the integral of the complex valued function independent of path and hence well defined as well. For this, consider the integral of $f$ over a smooth curve $\mathcal{C}$ in the $(x, y)$-plane,

$$\begin{aligned} \int_{\mathcal{C}} f(z)dz &= \int_{\mathcal{C}} (u+iv)(dx+idy) \qquad (24) \\ &= \int_{\mathcal{C}} udx - vdy + i\int_{\mathcal{C}} vdx + udy, \end{aligned}$$

where we have just used the complex multiplication to construct two real line integrals giving the real and imaginary part of $\int_{\mathcal{C}} f(z)dz$, written as integrals of 1-forms as in the third way to write line integrals in (8). Thus according to (8), the real and imaginary parts are line integrals for the two real vector fields $\mathbf{G}(x, y) = u(x, y)\mathbf{i} - v(x, y)\mathbf{j}$ and

$\mathbf{H}(x,y) = v(x,y)\mathbf{i} + u(x,y)\mathbf{j}$, respectively. It follows that

$$Curl\mathbf{G} = -v_x - u_y = 0,$$

and

$$Curl\mathbf{H} = u_x - v_y = 0,$$

which are both zero exactly by the Cauchy Riemann equations (23). Since the Curl's both vanish, it follows from Stokes Theorem that both integrals in (24) are independent of path. It is really quite remarkable: the Cauchy Riemann equations, derived to make the complex *derivative* well defined and independent of the limit $\Delta z \to 0$, turn out to also be the zero Curl conditions for guaranteeing the path independence of the two *integrals* that define the complex valued function which is the anti-derivative! However, we know we can only apply the zero Curl condition to get path independence in a simply connected region of the plane where we are guaranteed that a closed curve can be spanned by a surface with that curve as boundary, such that the Curl is zero on that surface.

We consider, finally, the case of an analytic function with a pole as a singularity. The simplest analytic function with a pole is the function $f(z) = 1/z$. It is not difficult to show that the complex derivative of $f(z)$ exists away from $z = 0$. Indeed,

$$\begin{aligned} f'(z) &= \lim_{\Delta z \to 0} \frac{1}{\Delta z} \left\{ \frac{1}{z + \Delta z} - \frac{1}{z} \right\} \\ &= \lim_{\Delta z \to 0} \frac{1}{\Delta z} \left\{ \frac{z - z - \Delta z}{(z + \Delta z) z} \right\} \\ &= -\frac{1}{z^2} \end{aligned}$$

where the limit exists and is independent of $\Delta z \to 0$. Thus the Cauchy-Riemann equations hold, and the complex line integrals of $\mathbf{G}$ and $\mathbf{H}$ are independent of path so long as we don't take a curve that encircles the singularity at $z = 0$. Indeed, we can calculate $\mathbf{G}$ and $\mathbf{H}$ directly in the case $f(z) = 1/z$:

$$\begin{aligned}\int_{\mathcal{C}} \frac{dz}{z} &= \int_{\mathcal{C}} \frac{dx + idy}{x + iy} = \int_{\mathcal{C}} \frac{(dx + idy)(x - iy)}{x^2 + y^2} \\ &= \int_{\mathcal{C}} \frac{xdx + ydy}{x^2 + y^2} + i \int_{\mathcal{C}} \frac{-ydx + xdy}{x^2 + y^2}.\end{aligned}$$

Thus in the case $f(z) = 1/z$ we have

$$\mathbf{G} = \frac{x}{r^2} + \frac{y}{r^2}, \qquad \mathbf{H} = \frac{-y}{r^2} + \frac{x}{r^2}.$$

Now both are Curl-free and both have a singularity at $r = 0$. Thus their line integrals will be the same on any simple closed path encircling the singularity once in the same direction. Moreover, our $\mathbf{G}$ and $\mathbf{H}$ turn out to be *exactly* the vector fields we introduced in (13) and (18). Thus for any such path $\mathcal{C}$, by (19) and (17) we have

$$\int_{\mathcal{C}} f(z)dz = \int_{\mathcal{C}} \frac{xdx + ydy}{x^2 + y^2} + i \int_{\mathcal{C}} \frac{-ydx + xdy}{x^2 + y^2} = 2\pi i.$$

In Complex Variables, $2\pi i$ is the factor that sits in front of the residue $a$ in a pole that looks like $f(z) = a/z$ near $z = 0$. One can show that $1/z$ is the only power, positive *or* negative, that contributes a nonzero residue[1], and an analytic function can be expanded in (inverse) powers of $z$ at a point singularity (based on its *Laurant* series), so we have established the essence of the Cauchy Residue Theorem: The line integral of a complex analytic function around a closed curve in the $(x, y)$-plane is exactly equal to $2\pi i$ times the sum of all the residues enclosed by the curve.

[1]Indeed, it is not difficult to prove that $\int \mathcal{C} f(z)dz = F(B) - F(A)$ for any curve $\mathcal{C}$ with endpoints $A$ and $B$, where $F$ is a complex antiderivative satisfying $F'(z) = f(z)$. Since $F(z) = z^{n+1}/n + 1$ is a single-valued antiderivative of $f(z) = 1/z^n$ for every positive and negative integer $n$ *except* $n = -1$, it follows that the integral of $f(z) = z^n$ around the singularity $z = 0$ is zero for all integers except $n = -1$.