VI The Cauchy-Goursat Theorem

So far we have:

- **Defn**: f is analytic if $f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$ exists.

Usually we speak of f being analytic in an open set or domain $D \subseteq \mathbb{C}$

Here: $f : D \to \mathbb{C}$ $\qquad w = f(z)$

$z \mapsto w \qquad u + iv = f(x+iy)$

Our starting point:

- **Theorem** (Cauchy-Riemann) f is analytic at $z_0 \in \mathbb{C}$ iff u and v are differentiable and satisfy the Cauchy-Riemann Equations

$$u_x = v_y$$
$$u_y = -v_x$$

(At this stage, the derivatives need not be continuous!)

- Recall: that it was not enough for C-R alone to hold in order to conclude $f'(z_0)$ exists – we needed u, v be differentiable at z_0 as well – C-R is a pointwise condition.

- Recall: u_x & u_y could exist at z_0 w/o u being differentiable – for differentiable we need that all directional derivatives exist i.e, $\nabla u \cdot \vec{v} = \frac{d}{dt} u(\underset{\sim}{x}_0 + t\vec{v})\Big|_{t=0}$, which is equiv to an approximating tangent plane at z_0.

- The most important theorem from Advanced Calculus used to establish at $u(x,y)$ is continuously differentiable in D is

Theorem (MAT 127) If u_x & u_y exist and are cont in D, then u is continuously diff in D.

(Hence $f'(z)$ is continuous.)

• Our extension of the Fundamental Thm of Calculus (FTC) based on Green's Thm required not only that f be analytic, but also required the assumption that $f'(z)$ be continuous. This is because we use the theorem that the Riemann Integral exists (the Riemann sums defining it converge) if the function is continuous

We proved:

Theorem (Weak form of Cauchy's Theorem)

If f is analytic (ie. $f'(z)$ exists) in an open set $D \subseteq \mathbb{C}$ s.t. D is simply connected and $f'(z)$ is continuous, then

(1) f has an anti-derivative F, $F'(z) = f(z)$

(2) The FTC holds $\int_{\mathcal{C}} f(z)dz = F(B) - F(A)$

(3) $\oint_{\mathcal{C}} f(z)dz = 0$

• Our question: do there exist "low regularity" analytic functions, by which we mean $f'(z)$ exists at every point in D, but f' is not a continuous function?

The answer is No! And the proof requires the Cauchy-Goursat Theorem.

The Cauchy-Goursat Theorem is a new starting point for Complex Variables, everything follows from this. I.e., for the logical development of the subject, everything we did based on Green's Thm to prove (1), (2), (3) can be thrown away. A graduate class on complex variables can <u>start</u> with Cauchy-Goursat, and develop everything from there. The problem is - w/o (1), (2), (3) its not clear how anyone could discover this!

• <u>Background</u>: (Math 127A) A function can have a derivative at every point without that derivative being cont.

HW #1 Prove that if $f: \mathbb{R} \to \mathbb{R}$ is differentiably in D (so $\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$ exists) then f is continuous.

HW #2 The following counterexample shows that $f'(x)$ can exist w/o f' being continuous. Prove $f(x) = x \sin \frac{1}{x}$ has a derivative at $x=0$, but $f'(x)$ is not continuous @ $x=0$.

(Its a subtle issue because f' won't have simple jump discontinuities when f' not cont)

- Recall where in the proof of (1), (2), (3) we required that u_x, u_y, v_x, v_y be continuous (i.e., f' is continuous).

Recall the Proof:

$$\oint_{\mathcal{C}} f(z)\,dz = \oint_{\mathcal{C}} (u+iv)(dx+idy)$$

$$= \oint_{\mathcal{C}} u\,dx - v\,dy + i\oint_{\mathcal{C}} v\,dx + u\,dy$$

$$= \oint_{\mathcal{C}} \vec{G}_1 \cdot \vec{T}\,ds + i\oint_{\mathcal{C}} \vec{G}_2 \cdot \vec{T}\,ds$$

$\vec{G}_1 = \overrightarrow{(u,-v)}$ $\qquad$ $\vec{G}_2 = \overrightarrow{(v,u)}$

Now C-R $\Leftrightarrow$ $\text{Curl}\,\vec{G}_1 = 0 = \text{Curl}\,\vec{G}_2$

Greens Thm:

$$\iint_A \text{Curl}\, G_i \, dA = \oint_C \vec{G}_i \cdot \vec{T} \, ds$$

$$\text{Curl}\,\vec{G}_i = 0 \Rightarrow \oint_C \vec{G}_i \cdot \vec{T} \, ds = 0$$

<u>But</u>: Pf of Gn's Thm Requires that all derivatives of $\vec{G}_i$ be continuous.

Eg $\vec{G} = M\,dx + N\,dy$

$$\iint_A N_x - M_y \, dA = \iint_A N_x \, dx\,dy - \iint_A M_y \, dy\,dx$$

To iterate the integrals require N_x & M_y be continuous — ow doesn't make sense as $\iint_A N_x \, dA$ not defined always when N_x discont

- The FTC then follows from Cauchy's thm that

$$\oint_{\mathcal{C}} f(z)\,dz = 0$$

I.e., $\vec{G}_1 = \nabla U$ by $U = \int_A^{(x,y)} \vec{G}_1 \cdot \vec{T}\,ds$ ← indep't of path

$\vec{G}_2 = \nabla V$ by $V = \int_A^{(x,y)} \vec{G}_2 \cdot \vec{T}\,ds$

$F(z) = U + iV$, $F'(z) = f(z)$ and

$$\int_{\mathcal{C}} f(z)\,dz = F(B) - F(A) \checkmark$$

<u>The point</u>: We only know functions f which solve the C-R equations <u>and</u> have continuous partial derivs of u, v, satisfy the Fundamental Thm of Calc.

2 The Cauchy-Goursat Theorem:

Theorem (Cauchy-Goursat) If f is differentiable in a s.c. domain $D \underset{\text{open}}{\subseteq} \mathbb{C}$, then

$$\oint_{\mathcal{C}} f(z)\,dz = 0 \qquad \text{(C-G)}$$

for every closed curve $\mathcal{C}$ in D.

Corollary (Cauchy Integral Formula)

Assume f is analytic in s.c. $D \underset{\text{open}}{\subseteq} \mathbb{C}$.

Then

$$f(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f(w)}{w-z}\,dw \qquad \text{(CIF)}$$

(Holds for any positively oriented simple closed curve in D.)

Note that neither requires the assumption that f' is continuous. Differentiating (CIF) thru $\int$-sign wrt z (OK since $\frac{\partial}{\partial z}\left(\frac{f(w)}{w-z}\right)$ is cont) we see that f has continuous deriv's of all orders

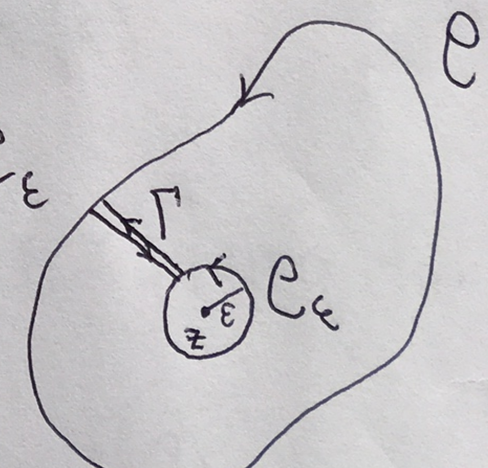
Proof of (CIF):

Assuming the Cauchy-Goursat Thm, f analytic $\Rightarrow$ $\dfrac{f(w)}{w-z}$ is an analytic function of w for every fixed z, so by (C-G)

$$\oint_{C} \frac{f(w)}{w-z}\,dw = 0$$

$\forall$ closed curve which does not contain z. Let C_ε be the boundary of the ball of radius ε center z. Then

$$0 = \underbrace{\int_{C-\Gamma-C_\varepsilon+\Gamma}}_{\text{closed curve which does not contain } z} \frac{f(w)}{w-z}\,dz = \int_{C} \frac{f(w)}{w-z} - \cancel{\int_{\Gamma}} + \cancel{\int_{\Gamma}} - \int_{C_\varepsilon}$$



so

$$\int_{C} \frac{f(w)}{w-z}\,dw = \int_{C_\varepsilon} \frac{f(w)}{w-z}\,dw$$

Now since f is cont, as $\varepsilon \to 0$, $f(w)$ tends to its value $f(z)$ for $w \in C_\varepsilon$. Thus by continuity and properties of integrals we have...

$$\int_{C_\varepsilon} \frac{f(w)}{w-z}\,dw = f(z)\underbrace{\int_{C_\varepsilon} \frac{1}{w-z}\,dw}_{=2\pi i} + o(\varepsilon)$$

little "oh" of ε tends to zero as $\varepsilon \to 0$

Thus

$$\int_{C} \frac{f(w)}{w-z}\,dw = 2\pi i f(z) + o(\varepsilon)$$

Taking $\varepsilon \to 0$ so $O(\varepsilon) \to 0$ & solving for $f(z)$ gives

$$f(z) = \frac{1}{2\pi i}\int_{C} \frac{f(w)}{w-z}\,dw$$

<u>Proof</u> of Cauchy Goursat Theorem:

We do the case $D = B_r(z_0) \subseteq \mathbb{C}$.
this contains all the ideas, and extension to any simply connected domain is intuitive.
There are two parts

(I) Main Lemma: If $\mathcal{C}$ is a rectangle $R \subseteq D$, then $\oint_R f(z)\,dz = 0$

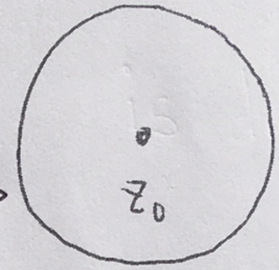
(II) Using (I) we construct an antideriv $F(z) = \int_{\mathcal{C}} f(z)\,dz$ where $\mathcal{C}$ is a curve which follows the sides of rectangles in D.

Once we get $F'(z) = f(z)$, we have FTC so

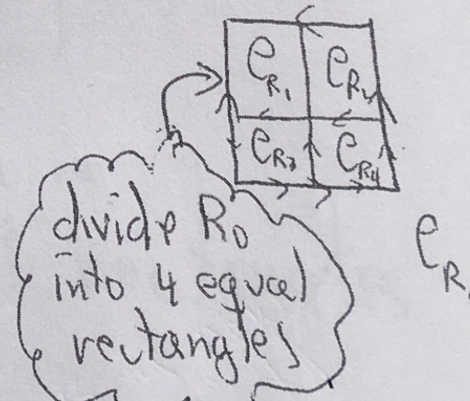
$$\oint_{\mathcal{C}} f(z)\,dz = F(B) - F(A) = 0 \;. \quad \checkmark$$
$$A = B$$

Proof: Part (I) for the case $D = B_r(z_0) \subseteq \mathbb{C}$.
This extends by same idea to s.c. regions.
[This is a famous proof]

Assume: f is analytic in $B_r(z_0)$. Our aim is prove $\oint_{\mathcal{C}} f(z)\,dz = 0$ $\forall$ closed $\mathcal{C} \subseteq B_r(z_0)$



Main Lemma: Assume $\mathcal{C}_{R_0}$ is a rectangle in $B_r(z_0)$. Then $\int_{\mathcal{C}_{R_0}} f(z)\,dz = 0$.



Pf $\int_{\mathcal{C}_{R_0}} f = \int_{\mathcal{C}^1_{R_0}} f + \int_{\mathcal{C}^2_{R_0}} f + \int_{\mathcal{C}^3_{R_0}} f + \int_{\mathcal{C}^4_{R_0}} f$.

Then $\left|\int_{\mathcal{C}_{R_0}} f\right| \leq 2^2 \left|\int_{\mathcal{C}_{R_1}} f\right|$ "for ~~some~~ R_2 chosen to be the biggest of $\left|\int_{\mathcal{C}^i_{R_0}} f(z)\,dz\right|$"

Now partition $\mathcal{C}_{R_1}$ into 4 equal rectangles

$$\mathcal{C}_{R_1^1} \quad \mathcal{C}_{R_1^2} \quad \mathcal{C}_{R_1^3} \quad \mathcal{C}_{R_1^4}$$

Then $\left|\int_{\mathcal{C}_{R_1}} f(z)\,dz\right| \le 2^2 \left|\int_{\mathcal{C}_{R_2}} f(z)\,dz\right|$

for R_2 chosen so $\left|\int_{\mathcal{C}_{R_1^i}} f(z)\,dz\right|$ is max.

Then

$$\left|\int_{\mathcal{C}_{R_0}} f\right| \le (2^2)^2 \left|\int_{\mathcal{C}_{R_2}} f\right|$$

... construct a nested sequ of rectangles $R_1 \supset R_2 \supset \cdots$ s.t.

$$\left|\int_{\mathcal{C}_{R_0}} f(z)\,dz\right| \le (4)^n \left|\int_{\mathcal{C}_{R_n}} f(z)\,dz\right|$$

But the perimiter of $\mathcal{C}_{R_n} = \frac{1}{2^n}$ Perimeter $(R_0) = \frac{P}{2^n}$

Now $\bigcap_{n=0}^{\infty} R_n$ = a single point z_0 (Nested Intervals Theorem). $f'(z_0)$ exists, so

$$f(z) - f(z_0) - f'(z_0)(z - z_0) = o(1)|z - z_0|$$

↑ little $o(1)$ tends to zero as $z \to z_0$

$$\therefore \left| \int_{\mathcal{C}_{R_n}} f(z)\,dz \right| = \left| \int_{\mathcal{C}_{R_n}} \underbrace{f(z_0) + f'(z_0)(z - z_0)} + o(1)|z - z_0|\,dz \right|$$

we know $\oint_{\mathcal{C}} f(z_0) + f'(z_0)(z - z_0)\,dz = 0$

$$\therefore \left| \int_{\mathcal{C}_{R_n}} f(z)\,dz \right| \le \underset{\uparrow\ |\mathcal{C}_{R_n}|}{\frac{P}{2^n}} \cdot \underset{\uparrow\ |z - z_0|}{\frac{P}{2^n}}\, o(1) = o(1)\,\frac{P^2}{4^n}$$

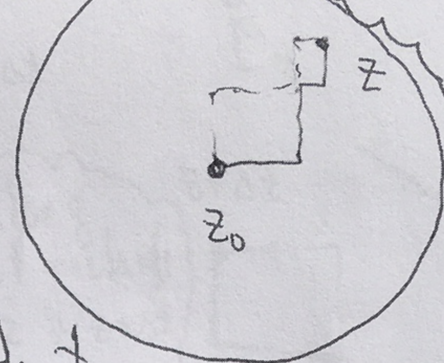
$$\therefore \left| \int_{\mathcal{C}_{R_0}} f(z)\,dz \right| \le 4^n \left| \int_{\mathcal{C}_{R_n}} f(z)\,dz \right| \le o(1)\, 4^n\, \frac{P^2}{4^n} \xrightarrow[n \to \infty]{} 0$$

$$\therefore \int_{\mathcal{C}_{R_0}} f(z)\,dz = 0 \ \checkmark$$

Proof: Part (II) Cauchy-Goursat Thm for $B_r(z_0)$. First note -

strategy - we find antideriv of $f(z)$ & use FTC to show $\oint_C f(z)\,dz = 0$

$\int_{C_z} f(z)\,dz$ is the same for every curve C_z taking $z_0 \to z$ along st lines that move along sides of rectangle inside $B_r(z_0)$.



Thus define -

$$F(z) = \int_{C_z} f(z)\,dz$$

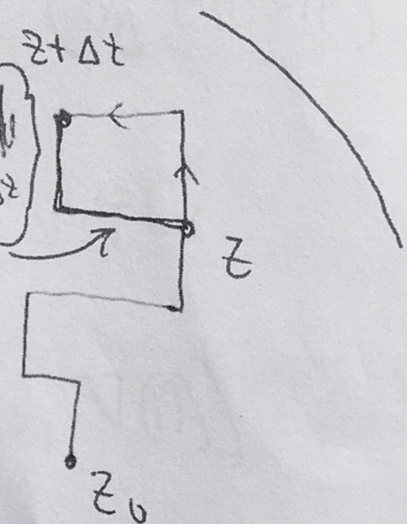
where C_z consists of pw smooth curve each piece being the side of a rectangle in $B_r(z_0)$.

Now consider

$$\frac{F(z+\Delta z)-F(z)}{\Delta z} = \left[\int_{C_{z+\Delta z}} f(\xi)\,d\xi - \int_{C_z} f(\xi)\,d\xi\right]\frac{1}{\Delta z}$$

$$= \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(\xi)\,d\xi$$

Curve along sides of rectangle having z & $z+\Delta z$ on opposite corners



But

$$\left|\frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z)\right|$$

$$= \left|\frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(\xi)\,d\xi - \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(z)\,d\xi\right|$$

This equals $f(z)$ as it is constant

$$= \left|\frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(\xi)-f(z)\;dz\right|$$

$$\leq \frac{1}{|\Delta z|}\int_{z}^{z+\Delta z} |f(\xi)-f(z)|\,dz \leq \frac{1}{|\Delta z|}\cdot|\Delta z|\max_{B_{2|\Delta z|}(z)}|f(\xi)-f(z)|$$

f continuous $\ni$ $\max_{B_{2|\Delta z|}(z)} |f(\xi)-f(z)| \longrightarrow 0$ as $\Delta z \to 0$ ✓

Conclude: $F(z)$ is an anti-deriv of $f(z)$.

Let

$$F(z) = U(x,y) + iV(x,y)$$

Then $\vec{G}_1 = \nabla U$ & $\vec{G}_2 = \nabla V$ (by C-R FIP)

$$\therefore \int_{\mathcal{C}} f(z)\,dz = \int_{\mathcal{C}} \nabla U \cdot \vec{T}\,ds + i\int_{\mathcal{C}} \nabla V \cdot \vec{T}\,ds$$

$$= U(B) - U(A) + i\big(V(B) - V(A)\big)$$

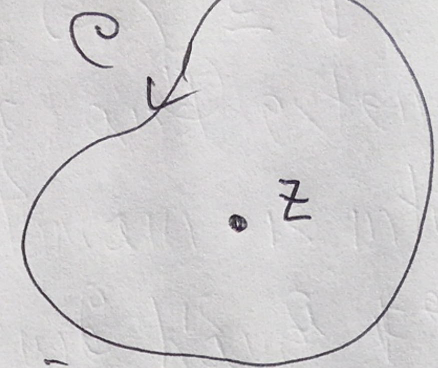
$$= F(B) - F(A)$$

for pw. smooth any curve $\mathcal{C} \subseteq B_r(z_0)$.

Thus: $\oint_{\mathcal{C}} f(z)\,dz = 0 \quad \forall$ closed curve in $B_r(z_0)$ as claimed. $\blacksquare$

☑ Cauchy Integral Formula:

We have:



$$f(z) = \frac{1}{2\pi i} \oint_{C} \frac{f(w)}{w-z}\, dz$$

Since $g(w,z) = \frac{f(w)}{w-z}$ satisfies $\frac{\partial g}{\partial z}$ is cont along C, we can diff thru $\int$-sign wrt z (by methods of advanced Calc) to obtain

$$f'(z) = \frac{1}{2\pi i} \oint_{C} \frac{f(w)}{(w-z)^2}\, dw$$

$\vdots$

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_{C} \frac{f(w)}{(w-z)^{n+1}}\, dw$$

so: f analytic $\Rightarrow$ $f^{(n)}(z)$ exists and is cont $\forall\, n \in \mathbb{N}$.