MIDTERM EXAM II
Math 21B
Temple-S05

- Print your name, section number and put your signature on the upper right-hand corner of this exam.

- Write only on the exam.

- Show all of your work, and justify your answers for full credit.

SCORES

#1

#2

#3

#4

TOTAL:
Problem 1. (18pts) Let $R$ be the region bounded by $y = x + x^3$, $x = 0$, and $x = 1$.

(a) (6pts) Find the area of the region $R$.

**Solution:** Area = \( \int_0^1 (x + x^3) \, dx = \left[ \frac{x^2}{2} + \frac{x^4}{4} \right]_0^1 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \)

(b) (6pts) Use the shell method to find the volume of the solid produced by revolving $R$ about the $y$-axis.

**Solution:** Partition $[0, 1]$ by $0 = x_0 < x_1 < \cdots < x_n = 1$ with $\Delta x = 1/n$ and $c_i = x_i$. Then the volume of the shell $\Delta V_i$ obtained by revolving the $i$'th rectangle around the $y$-axis is $\Delta V_i = 2\pi x_i (x_i + x_i^3) \Delta x$, so the total volume $V$ satisfies

\[
V \approx \sum_{i=1}^n 2\pi x_i (x_i + x_i^3) \Delta x \to \int_0^1 2\pi (x^2 + x^3) \, dx = 2\pi \left[ \frac{x^3}{3} + \frac{x^5}{5} \right]_0^1 = \frac{16\pi}{15},
\]

which gives the exact volume $V$ in the limit $\Delta x \to 0$.

(c) (6pts) Use the disk method to find the volume of the solid produced by revolving $R$ about the $x$-axis.

**Solution:** Same partition but $\Delta V_i = \pi (x_i + x_i^3)^2 \Delta x$, and

\[
V = \int_0^1 \pi (x^2 + x^3)^2 \, dx = \pi \int_0^1 [x^2 + 2x^4 + x^6] \, dx = \pi \left[ \frac{x^3}{3} + 2\frac{x^5}{5} + \frac{x^7}{7} \right]_0^1 = \pi \left( \frac{1}{3} + \frac{2}{5} + \frac{1}{7} \right) = \frac{92\pi}{105}.
\]
Problem 2. (30pts) The goal of this problem is to find \( \int \frac{x^4+3x^3+2}{x^2-4} \, dx \).

Do each of the following steps which accomplishes this:

(a) (8pts) Write the integrand as a polynomial \( q(x) \) plus a proper rational function \( \frac{r(x)}{p(x)} \).

Solution: By long division we get \( x^4 + 3x^3 + 2 = (x^2 - 4)(x^2 + 3x + 4) + (12x + 18) \), so \( \frac{x^4+3x^3+2}{x^2-4} = x^2 + 3x + 4 + \frac{12x+18}{x^2-4} \), so \( q(x) = x^2 + 3x + 4 \) and \( \frac{r(x)}{p(x)} = \frac{12x+18}{x^2-4} \).

(b) (6pts) Write \( \frac{r(x)}{p(x)} \) in terms of partial fractions. (Hint: remember to factor the denominator.)

Solution: \( \frac{r(x)}{p(x)} = \frac{12x+18}{x^2-4} = \frac{A}{x-2} + \frac{B}{x+2} \) where, (multiplying by \( x - 2 \) and setting \( x = 2 \)), \( A = \frac{21}{2} \), and similarly, \( B = \frac{3}{2} \).

(c) (6pts for (c) and (d)) Find the integral of \( q(x) \) and each partial fraction.

Solution: \( \int q(x) \, dx = \int (x^2 + 3x + 2) \, dx = \frac{x^3}{3} + \frac{3x^2}{2} + 4x + \text{const} \), \( \int \frac{A}{x-2} \, dx = A \ln(x-2) + \text{const} \), and \( \int \frac{B}{x+2} \, dx = B \ln(x+2) + \text{const} \).

(d) Write the final formula for the integral: \( \int \frac{x^4+3x^3+2}{x^2-4} \, dx = \) \( \int \left( q(x) + \frac{r(x)}{p(x)} \right) \, dx = \frac{x^3}{3} + \frac{3x^2}{2} + 4x + \frac{21}{2} \ln(x-2) + \frac{3}{2} \ln(x+2) + \text{const} \).
(e) (10pts) Letting capital letters denote independent constants, write the correct form of the partial-fraction expansion of \( \frac{x^2-1}{(x^2-4)(x-2)(x^2+2x+3)^2} \)

**Solution:** First, \( x^2-4 \) is not irreducible, so write \( \frac{x^2-1}{(x^2-4)(x-2)(x^2+2x+3)^2} = \frac{A}{x^2-4} + \frac{B}{x-2} + \frac{Cx+D}{x^2+2x+3} + \frac{Ex+F}{(x^2+2x+3)^2} \)

**Problem 3. (16pts)** Recall that the equation for an ellipse is \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \). Find the volume \( V \) of the ellipsoid obtained by rotating the ellipse about the \( x \)-axis.

**Solution:** Using the disk method leads to \( V = \int_{-a}^{a} \pi r^2 dx \), where \( r = y = b\sqrt{1 - \frac{x^2}{a^2}} \). Thus \( V = \int_{-a}^{a} \pi b^2 \left(1 - \frac{x^2}{a^2}\right) dx = \pi b^2 \left[x - \frac{x^3}{3a^2}\right]_{-a}^{a} = \frac{4\pi}{3}ab^2 \). Note that, like the sphere, the volume of the ellipsoid is given by the easy to remember formula

\[
V = \frac{4\pi}{3} \times \text{width} \times \text{height} \times \text{depth} \]
Problem 4a. (12pts) Evaluate $\int \frac{1}{x^2 + 2x + 2} \, dx$.

Solution: The denominator is an irreducible quadratic, so completing the square reduces it to an inverse tangent:

$$\int \frac{1}{x^2 + 2x + 2} \, dx = \int \frac{1}{(x+1)^2 + 1} \, dx = \tan^{-1}(x+2) + \text{const}$$

Problem 4b. (12pts) Starting with the identity $\cos 2x = \cos^2 x - \sin^2 x$, derive the trigonometric identity that enables you to solve $\int \sin^2 x \, dx$, and then use it to get the solution.

Solution: $\cos^2 x - \sin^2 x = 1 - 2\sin^2 x = \cos 2x$, so $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$. Using this we obtain $\int \sin^2 x \, dx = \frac{x}{2} - \frac{\sin 2x}{4} + \text{const}$.

Problem 4c. (12pts) Reduce the integral $\int \sqrt{x^2 - 1} \, dx$ to the integral of a rational function of $u$, (to which the method of partial fractions applies.) You may use the fact that the substitution $u = \tan(\theta/2)$ leads to $\cos \theta = \frac{1-u^2}{1+u^2}$, $\sin \theta = \frac{2u}{1+u^2}$, $d\theta = \frac{2du}{1+u^2}$.

Solution: Let $x = \sec \theta$, $dx = \sec \theta \tan \theta$. Thus $\int \sqrt{\sec^2 \theta - 1} \, \sec \theta \tan \theta \, d\theta = \int \sec^2 \theta \, \sec \theta \, d\theta = \int \frac{\sin^2 \theta}{\cos^3 \theta} \, d\theta$, which by the substitution $u = \tan(\theta/2)$, reduces to $\int \frac{8u^2 du}{(1-u^2)^3}$. 