MIDTERM EXAM II
Math 21B
Temple-S05

−Print your name, section number and put your signature on the upper right-hand corner of this exam.

−Write only on the exam.

−Show all of your work, and justify your answers for full credit.

SCORES

#1

#2

#3

#4

TOTAL:

Problem 1. (18pts) Let **R** be the region bounded by $y = x + x^3$, $x = 0$, and $x = 1$.

(a) (6pts) Find the area of the region **R**.

Solution: Area$=\int_0^1 (x + x^3)dx = \left[\frac{x^2}{2} + \frac{x^4}{4}\right]_0^1 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

(b) (6pts) Use the shell method to find the volume of the solid produced by revolving **R** about the y-axis.

Solution: Partition $[0, 1]$ by $0 = x_0 < x_1 < \cdots < x_n = 1$ with $\Delta x = 1/n$ and $c_i = x_i$. Then the volume of the shell ΔV_i obtained by revolving the $i'th$ rectangle around the y-axis is $\Delta V_i = 2\pi x_i(x_i + x_i^3)\Delta x$, so the total volume V satisfies

$$V \approx \sum_{i=1}^{n} 2\pi x_i(x_i{+}x_i^4)\Delta x \to \int_0^1 2\pi(x^2{+}x^3)dx = 2\pi\left[\frac{x^3}{3} + \frac{x^5}{5}\right]_0^1 = \frac{16\pi}{15},$$

which gives the exact volume V in the limit $\Delta x \to 0$.

(c) (6pts) Use the disk method to find the volume of the solid produced by revolving **R** about the x-axis.

Solution: Same partition but $\Delta V_i = \pi(x_i + x_i^3)^2\Delta x$, and

$$\begin{aligned} V &= \int_0^1 \pi(x^2 + x^3)^2 dx = \pi\int_0^1 \left[x^2 + 2x^4 + x^6\right] \\ &= \pi\left[\frac{x^3}{3} + 2\frac{x^5}{5} + \frac{x^7}{7}\right]_0^1 = \pi\left(\frac{1}{3} + \frac{2}{5} + \frac{1}{7}\right) = \frac{92\pi}{105}. \end{aligned}$$

Problem 2. (30pts) The goal of this problem is to find $\int \frac{x^4+3x^3+2}{x^2-4}dx$. Do each of the following steps which accomplishes this:

(a) (8pts Write the integrand as a polynomial $q(x)$ plus a proper rational function $\frac{r(x)}{p(x)}$.

Solution: By long division we get $x^4 + 3x^3 + 2 = (x^2 - 4)(x^2 + 3x + 4) + (12x + 18)$, so $\frac{x^4+3x^3+2}{x^2-4} = x^2 + 3x + 4 + \frac{12x+18}{x^2-4}$, so $q(x) = x^2 + 3x + 4$ and $\frac{r(x)}{p(x)} = \frac{12x+18}{x^2-4}$.

(b) (6pts) Write $\frac{r(x)}{p(x)}$ in terms of partial fractions. (Hint: remember to factor the denominator.)

Solution: $\frac{r(x)}{p(x)} = \frac{12x+18}{x^2-4} = \frac{A}{x-2} + \frac{B}{x+2}$ where, (multiplying by $x - 2$ and setting $x = 2$), $A = \frac{21}{2}$, and similarly, $B = \frac{3}{2}$.

(c) (6pts for (c) and (d)) Find the integral of $q(x)$ and each partial fraction.

Solution: $\int q(x)dx = \int (x^2+3x+2)dx = \frac{x^3}{3} + \frac{3x^2}{2} + 4x + const$, $\int \frac{A}{x-2}dx = A\ln(x-2) + const$, and $\int \frac{B}{x+2}dx = B\ln(x+2) + const$.

(d) Write the final formula for the integral: $\int \frac{x^4+3x^3+2}{x^2-4}dx =$

Solution: $\int \frac{x^4+3x^3+2}{x^2-4}dx = \int \left(q(x) + \frac{r(x)}{p(x)}\right) dx = \frac{x^3}{3} + \frac{3x^2}{2} + 4x + \frac{21}{2}\ln(x-2) + \frac{3}{2}\ln(x+2) + const.$

(e) (10pts) Letting capital letters denote independent constants, write the correct form of the partial-fraction expansion of $\frac{x^2-1}{(x^2-4)(x-2)(x^2+2x+3)^2}$

Solution: First, x^2-4 is not irreducible, so write $\frac{x^2-1}{(x^2-4)(x-2)(x^2+2x+3)^2} = \frac{x^2-1}{(x+2)(x-2)^2(x^2+2x+3)^2}$ which has partial fraction expansion $\frac{A}{x+2} + \frac{B}{x-2} + \frac{C}{(x-2)^2} + \frac{Dx+E}{x^2+2x+3} + \frac{Fx+G}{(x^2+2x+3)^2}$

Problem 3. (16pts) Recall that the equation for an ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Find the volume V of the ellipsoid obtained by rotating the ellipse about the x-axis.

Solution: Using the disk method leads to $V = \int_{-a}^{a} \pi \; r^2 dx$, where $r = y = b\sqrt{1 - \frac{x}{a^2}}$. Thus $V = \int_{-a}^{a} \pi b^2 \left(1 - \frac{x^2}{a^2}\right) dx = \pi b^2 \left[x - \frac{x^3}{3a^2}\right]_{-a}^{a} = \frac{4\pi}{3} ab^2$. Note that, like the sphere, the volume of the ellipsoid is given by the easy to remember formula

$$V = \frac{4\pi}{3} \times \frac{width}{2} \times \frac{height}{2} \times \frac{depth}{2}.$$

Problem 4a. (12pts) Evaluate $\int \frac{1}{x^2+2x+2}\ dx$.

Solution: The denominator is an irreducible quadratic, so completing the square reduces it to an inverse tangent: $\int \frac{1}{x^2+2x+2}\ dx = \int \frac{1}{x^2+2x+1-1+2}\ dx = \int \frac{1}{(x+2)^2+1}\ dx = tan^{-1}(x+2) + const$

Problem 4b. (12pts) Starting with the identity $cos2x = cos^2x - sin^2x$, derive the trigonomentric identity that enables you to solve $\int sin^2\ dx$, and then use it to get the solution.

Solution: $cos^2x - sin^2x = 1 - 2sin^2x = cos2x$, so $sin^2x = \frac{1}{2}(1 - cos2x)$. Using this we obtain $\int sin^2x\ dx = \frac{x}{2} - \frac{sin2x}{4} + const$.

Problem 4c. (12pts) Reduce the integral $\int \sqrt{x^2-1}\ dx$ to the integral of a rational function of u, (to which the method of partial fractions applies.) You may use the fact that the substitution $u = tan(\theta/2)$ leads to $cos\theta = \frac{1-u^2}{1+u^2}$, $sin\theta = \frac{2u}{1+u^2}$, $d\theta = \frac{2du}{1+u^2}$.

Solution: Let $x = sec\theta$, $dx = sec\theta\ tan\theta$. Thus $\int \sqrt{sec^2\theta - 1} sec\theta\ tan\theta\ d\theta = \int tan^2\theta\ \ sec\theta\ \ d\theta = \int \frac{sin^2\theta}{cos^3\theta}\ d\theta$, which by the substitution $u = tan(\theta/2)$, reduces to $\int \frac{8u^2du}{(1-u^2)^3}$.