Solutions

Name: ____________________

Student ID#: ____________________

Section: ____________________

Midterm Exam 2

Wednesday, March 4

MAT 21C, Temple, Winter 2020

Print names and ID's clearly, and have your student ID ready to be checked when you turn in your exam. Write the solutions clearly and legibly. Do not write near the edge of the paper or the stapled corner. Show your work on every problem. Correct answers with no supporting work will not receive full credit. Be organized and use notation appropriately. No calculators, notes, books, cellphones, etc. may be used on this exam.

Problem	Your Score	Maximum Score
1		28
2		25
3		16
4		16
5		15
Total		100

Problem #1 (28pts): Let $\mathbf{A} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{B} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$ be vectors.

(a) Find $\mathbf{A} \cdot \mathbf{B}$.

$$\overrightarrow{(2,-1,1)} \cdot \overrightarrow{(1,1,2)} = 2\cdot 1 + (-1)\cdot 1 + 1\cdot 2 = 3$$

(b) Find $\mathbf{A} \times \mathbf{B}$.

$$\vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ 2 & -1 & 1 \\ 1 & 1 & 2 \end{vmatrix} = \underset{\sim}{i}(-2-1) - \underset{\sim}{j}(4-1) + \underset{\sim}{k}(2+1)$$

$$= -3\underset{\sim}{i} - 3\underset{\sim}{j} + 3\underset{\sim}{k}$$

(c) Find a unit vector $\mathbf{u}$ in direction of $\mathbf{A}$.

$$\vec{u} = \frac{\vec{A}}{|\vec{A}|} = \frac{\overrightarrow{(2,-1,1)}}{\sqrt{2^2+(-1)^2+1^2}} = \frac{1}{\sqrt{6}} \overrightarrow{(2,-1,1)}$$

(d) Find a vector orthogonal to both $\mathbf{A}$ and $\mathbf{B}$.

$$\vec{n} = \vec{A} \times \vec{B} = \overrightarrow{(-3,-3,3)} = 3\overrightarrow{(-1,-1,1)}$$

(e) Find the area of the parallelogram spanned by **A** and **B**.

$$\text{Area} = |\vec{A} \times \vec{B}| = |3\,\overrightarrow{(-1,-1,1)}| = 3\sqrt{(-1)^2+(-1)^2+1^2}$$

$$= 3\sqrt{3}$$

(f) Find $Proj_{\mathbf{B}}\mathbf{A}$.

$$Proj_B A = \left(\vec{A} \cdot \frac{\vec{B}}{|\vec{B}|}\right)\frac{\vec{B}}{|\vec{B}|} \qquad |\vec{B}| = \sqrt{1^2+1^2+2^2} = \sqrt{6}$$

$$= \frac{1}{6}(\vec{A}\cdot\vec{B})\vec{B} = \frac{1}{6}\cdot 3\vec{B} = \frac{1}{2}\vec{B} = \frac{1}{2}\overrightarrow{(1,1,2)}$$

(g) Find the cosine of the angle between **A** and **B**.

$$\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta$$

$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|} = \frac{1}{6}\cdot 3 = \frac{1}{2}$$

(e) Find the equation of the plane tangent at P_0 to the constant temperature surface through P_0.

$$\overrightarrow{P_0P} \cdot \vec{n} = 0 \qquad \vec{n} = \nabla f(P_0)$$

$$\overrightarrow{(x-2, y+1, z-1)} \cdot \overrightarrow{(-1, 2, -4)} = 0$$

$$-x + 2 + 2y + 2 - 4z + 4 = 0$$

$$\boxed{-x + 2y - 4z = -2 - 2 - 4 = -8}$$

Problem #3 (16pts): Let $\mathbf{r}(t)$ denote the position vector of a particle at time t. Prove that if $|\mathbf{r}(t)| = \textit{const}$, then $\mathbf{r}'(\mathbf{t})$ is perpendicular to $\mathbf{r}(t)$ at every t.

Assume $|\vec{r}(t)| = const$

Then $|\vec{r}(t)|^2 = \vec{r}(t) \cdot \vec{r}(t) = const.$

Thus
$$0 = \frac{d}{dt} \vec{r}(t) \cdot \vec{r}(t)$$
$$= \vec{r}\,'(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}\,'(t)$$
$$= 2 \cdot \vec{r}\,'(t) \cdot \vec{r}(t)$$

Thus $\vec{r}\,'(t) \cdot \vec{r}(t) = 0 \Rightarrow \vec{r}\,'(t) \perp \vec{r}(t)$ ✓

Problem #2 (25pts): Let $w = f(x, y.z) = xyz^2$ give the temperature at point (x, y, z).

(a) Find $\nabla f(x, y, z)$. $\nabla f = \overrightarrow{(f_x, f_y, f_z)}$

$$\nabla f = \overrightarrow{(yz^2, xz^2, 2xyz)}$$

(b) Find the direction of steepest increase of temperature at point $P_0 = (2, -1, 1)$.

$$\nabla f(2,-1,1) = \overrightarrow{((-1)(1)^2, 2\cdot(1)^2, 2\cdot 2(-1)(1))}$$

$$= \overrightarrow{(-1, 2, -4)}$$

(c) Find $\frac{dw}{ds}$ in direction of steepest increase of temperature at P_0.

$$\frac{dw}{ds} = |\nabla f(2,-1,1)| = \sqrt{(-1)^2 + 2^2 + (-4)^2}$$

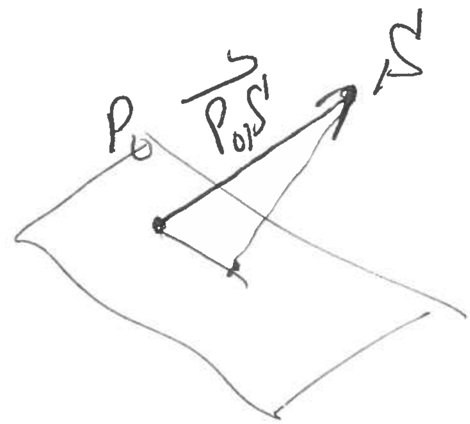
$$= \sqrt{21}$$

(d) Find $\frac{dw}{ds}$ in direction $\mathbf{v} = \mathbf{i} + 2\mathbf{j}$ at P_0.

$$\frac{dw}{ds} = \nabla f \cdot \frac{\vec{V}}{|\vec{V}|} = \overrightarrow{(-1, 2, -4)} \cdot \frac{\overrightarrow{(1, 2, 0)}}{\sqrt{1^2+2^2}} = \frac{-1+4}{\sqrt{5}} = \frac{3}{\sqrt{5}}$$

Problem #4 (16pts): Let $S = (1, 2, -1)$, and consider the general equation for a plane $ax + by - z = d$ where a, b, d are constants, possibly zero. Find a vector normal to the plane, and without allowing the possibility of dividing by zero, find a point on the plane. Use these to *derive* a general formula involving a, b, d for the distance from the point S to the plane. Draw a picture.

Point $P_0 = (0,0,-d)$



$\vec{N} = \overrightarrow{(a,b,-1)}$

$$\text{dist} = \left|\text{Proj}_{\vec{N}} \overrightarrow{P_0S}\right|$$

$$\vec{n} = \frac{\vec{N}}{|\vec{N}|} = \frac{\overrightarrow{(a,b,-1)}}{\sqrt{a^2+b^2+1}}$$

$$\text{dist} = \left|\text{Proj}_{\vec{n}} \overrightarrow{P_0S}\right| = \left|\vec{n} \cdot \overrightarrow{P_0S}\right|$$

$$\left|\overrightarrow{P_0S}\right| = \left|\overrightarrow{(1,2,-1+d)}\right|$$

$$\underbrace{\left|\vec{n} \cdot \overrightarrow{P_0S}\right|}_{\text{dist}} = \left|\frac{\overrightarrow{(a,b,-1)}}{\sqrt{a^2+b^2+1}} \cdot \overrightarrow{(1,2,-1+d)}\right| = \boxed{\left|\frac{a+2b-d+1}{\sqrt{a^2+b^2+1}}\right|}$$

Problem #5 (15pts): Assume each planet moves in uniform circular motion around the sun, (not a terrible approximation). Then taking the sun to be at the center, the trajectory of a planet is given by

$$\mathbf{r}(t) = R\{\cos\omega t + i\sin\omega t\}, \qquad (*)$$

where R is the radius of the planet's orbit, and ω adjusts its speed so that $\omega T = 2\pi$ where T is the time of one orbit. Kepler's third law states that

$$\frac{R^3}{T^2} = H$$

where H is a constant, the same for every planet. Verify Newton's inverse square force law by finding G, depending only on H, such that for each orbit $\mathbf{r}(t)$, $\mathbf{a}(t) = \mathbf{r}''(t)$ satisfies

$$\mathbf{a}(t) = -G\frac{1}{R^3}\mathbf{r}(\mathbf{t}).$$

$\vec{a} = \vec{r}\,''(t) = -\omega^2 \vec{r}(t)$ (by diff * twice)

But $\omega = \frac{2\pi}{T}$ so

$$\vec{a} = -\frac{4\pi^2}{T^2}\vec{r}(t)$$

But $\frac{R^3}{T^2} = H$ so

$$\vec{a} = -\frac{4\pi^2 R^3}{T^2}\frac{\vec{r}(t)}{R^3} = -\underbrace{4\pi^2 H}_{G}\frac{\vec{r}(t)}{R^3}$$