$\boxed{\text{Continuity}}$ Math 21C Temple

- Recall: $w = f(x, y, z)$

Think: w = temperature $\theta(x, y, z)$

We defined:

$$\nabla f = \overrightarrow{(f_x, f_y, f_z)} = \nabla f(x, y, z)$$

$$f_x = \frac{\partial f}{\partial x}(x, y, z) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y, z) - f(x, y, z)}{\Delta x}$$

"Diff wrt x holding y & z fixed as const"

$$f(x, y, z) = x^2y - yz$$

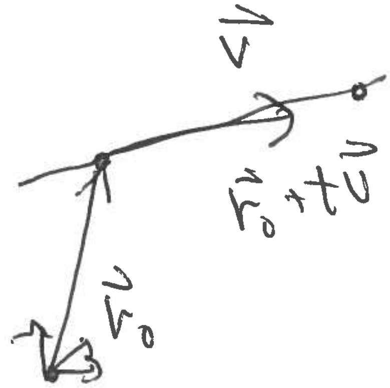
$$\frac{\partial f}{\partial x} = 2xy \quad \frac{\partial f}{\partial y} = x^2 - z \quad \frac{\partial f}{\partial z} = -y$$

$$\nabla f(x, y, z) = \overrightarrow{(2xy, x^2 - z, -y)}$$

- The importance of ∇f from Chain Rule

$$\frac{d}{dt} f(\vec{r}(t)) = \nabla f \cdot \vec{r}\,'(t)$$

<u>line</u>: $\vec{r}(t) = \vec{r}_0 + t\vec{v}$



$$\vec{r}\,'(t) = \vec{v}$$

$$|\vec{r}\,'(t)| = \frac{ds}{dt} = |\vec{v}|$$

<u>Conclude</u>: $\vec{v}$ <u>unit</u> $\Rightarrow ds = dt$

$$\therefore \frac{df}{ds} = \nabla f \cdot \frac{\vec{v}}{|\vec{v}|} = \text{"rate of change of temp wrt arclength in direction } \vec{v} \text{ from } \vec{r}_0\text{"}$$

- Since $\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \ldots$

we need to talk about <u>limits</u>

<u>**Defn**</u>: We say f is <u>continuous</u> at $\underset{\sim}{x}_0 = (x_0, y_0, z_0)$ if $\lim_{\underset{\sim}{x} \to \underset{\sim}{x}_0} f(x,y,z) = f(x_0, y_0, z_0)$

Words: "The graph of f is unbroken at $\underset{\sim}{x}_0$"

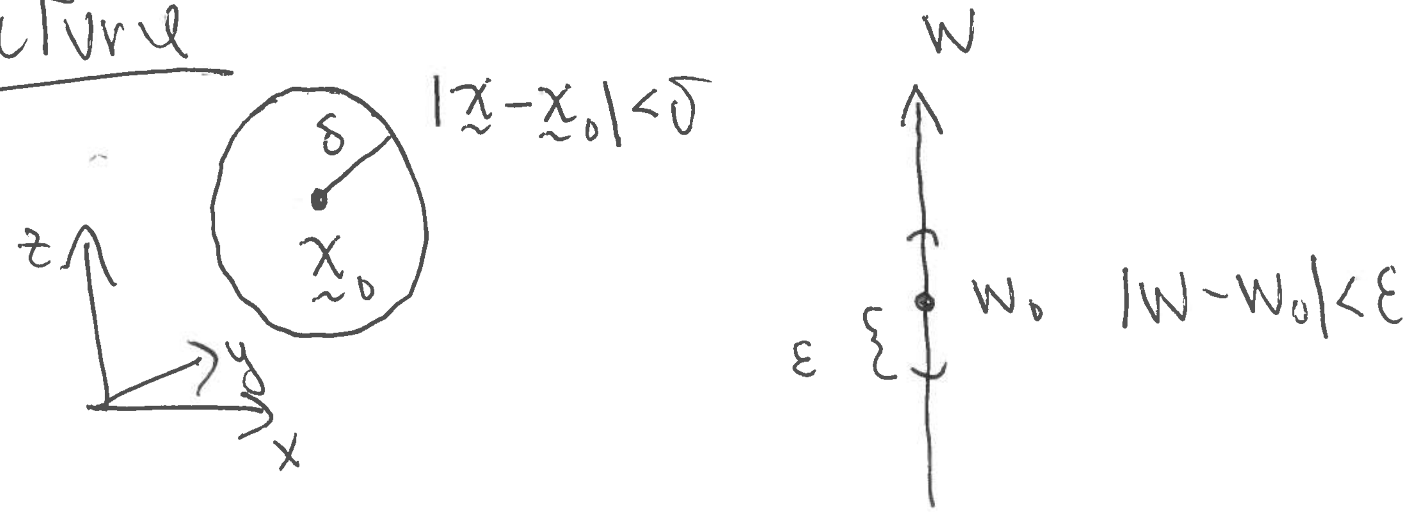
We need to define limits in $\mathbb{R}^3$

Defn: $\lim_{\underset{\sim}{x} \to \underset{\sim}{x}_0} f(\underset{\sim}{x}) = L$ if

Words: "The outputs of f get arbitrarily close to L if the inputs $\underset{\sim}{x}$ get sufficiently close to $\underset{\sim}{x}_0$"

Mathematically: $\forall\, \varepsilon > 0\ \exists\, \delta > 0$ st if $|\underset{\sim}{x} - \underset{\sim}{x}_0| < \delta$ then $|f(x,y,z) - L| < \varepsilon$.

Picture



inputs $\underset{\sim}{x}$ $\qquad$ w = output (temperature)

Example: Prove: $f(x,y,z) = xy^2 - z$

is continuous at $\underset{\sim}{x}_0 = \underset{\sim}{0} = (0,0,0)$

Note: $f(0,0,0) = 0$ so $L = 0$.

We must prove $\lim_{\underset{\sim}{x} \to \underset{\sim}{x}_0} f(\underset{\sim}{x}) = L = 0$.

Proof: To Prove: $\lim_{\underset{\sim}{x} \to \underset{\sim}{0}} f(\underset{\sim}{x}) = 0$

Fix $\varepsilon > 0$. We find $\delta > 0$ such that

if $|\underset{\sim}{x} - \underbrace{\underset{\sim}{0}}_{\underset{\sim}{x}_0}| < \delta$ then $|f(\underset{\sim}{x}) - \underbrace{0}_{L}| < \varepsilon$

We find δ st if $|\underset{\sim}{x}| < \delta$ then $|xy^2 - z| < \varepsilon$

Need: $|f(\underset{\sim}{x})| < \varepsilon \iff |xy^2 - z| < \varepsilon$

Suffices to make $|xy^2| < \frac{\varepsilon}{2}$ & $|z| < \frac{\varepsilon}{2}$

Note: $|x| < |(x,y,z)| = \sqrt{x^2+y^2+z^2}$

$|y| < |\underset{\sim}{x}| \qquad |z| < |\underset{\sim}{x}|$

So if $|x| < \frac{\varepsilon}{2}$ & $|y| < \frac{\varepsilon}{2}$ & $|z| < \frac{\varepsilon}{2}$

then $|xy^2| < \left|\frac{\varepsilon}{2}\left(\frac{\varepsilon}{2}\right)^2\right| < \varepsilon/2$, $|z| < \frac{\varepsilon}{2}$

$(\varepsilon < 1)$

$\therefore$ it suffies to make $|\underset{\sim}{x}| < \frac{\varepsilon}{2}$

End of thinking

Choose $\delta = \frac{\varepsilon}{2}$. Then if $|\underline{x}| < \frac{\varepsilon}{2}$,

also $|x| < \frac{\varepsilon}{2}$, $|y| < \frac{\varepsilon}{2}$, $|z| < \frac{\varepsilon}{2}$, so

$$|xy^2 - z| < |x||y|^2 + |z| < \frac{\varepsilon}{2}\left(\frac{\varepsilon}{2}\right)^2 + \frac{\varepsilon}{2} < \varepsilon \checkmark$$

Example: Show that if

$$w = f(x,y,z) = \frac{2xy + yz}{x^2+y^2+z^2} \qquad \underset{\sim}{x} \neq 0$$

then there is no way to define f at $\underset{\sim}{x} = 0$ so that f is cont.

Soln: To be cont, w has to have the same limit every way $\underset{\sim}{x} \to 0$.

Define $\underset{\sim}{x} = \vec{r}(t) = t\vec{v} \qquad \vec{v} = \overrightarrow{(a,b,c)}$ arbitrary

$$\boxed{\cancel{\lim_{t\to\infty} f(\vec{r}(t)) = \lim_{t\to\infty} f(t\vec{v})}} \qquad \lim_{t\to\infty} \vec{r}(t) = 0$$

$$f(\vec{r}(t)) = f(t\vec{v}) = \frac{2\cancel{t^2}ab + \cancel{t^2}bc}{\cancel{t^2}(a^2+b^2+c^2)} = \frac{2ab+bc}{a^2+b^2+c^2}$$

$$\therefore \lim_{t\to\infty} \underbrace{f(\vec{r}(t))}_{\text{tends to zero}} = \frac{2ab+bc}{a^2+b^2+c^2} \quad \text{depends on } \vec{v}$$

$\Rightarrow$ not unique ✓

☐ <u>Theorem</u>: $\lim_{\underset{\sim}{x} \to \underset{\sim}{x}_0} f(\underset{\sim}{x}) = L$ iff for every sequence $\underset{\sim}{x}_n \longrightarrow \underset{\sim}{x}_0$ we have

$$\lim_{n \to \infty} f(\underset{\sim}{x}_n) = f(\underset{\sim}{x}_0)$$

Recall: $\lim_{n \to \infty} \underset{\sim}{x}_n = \underset{\sim}{x}_0$ if $\forall \varepsilon \; \exists N$ st if $n > N$ then $|\underset{\sim}{x}_n - \underset{\sim}{x}_0| < \varepsilon$.

☐ ~~Theorem: if $f(x,y,z)$ is differentiable~~

We'd like to say that ~~a~~ ~~f~~ function is differentiable @ (x,y,z) then it is continuous @ (x,y,z).

Its not enough to have f_x, f_y, f_z exist at (x,y,z). For example

$$f(x,y) = \frac{xy}{x^2+y^2} \qquad f(0,0)=0$$

Is not cont. at $(0,0)$ but f_x & f_y do exist!

$$\frac{\partial f}{\partial x}(0,0) = \cancel{\frac{(x+\Delta x)y + xy}{(x+\Delta x)^2}} \quad \frac{f(\cancel{x}0+\Delta x, 0) - \overset{\to 0}{\cancel{f(0,0)}}}{\Delta x}$$

$$= \frac{\Delta x \cdot 0}{\Delta x^2 \cdot \Delta x} = 0 . \qquad \frac{\partial f}{\partial y}(0,0) = 0$$

<u>Defn</u>: f diff @ $(\underset{\sim}{x}_0)$ if $f_x\ f_y\ f_z$ exist <u>and</u>

$$\Delta w = f_x \Delta x + f_y \Delta y + f_z \Delta z +$$

$$\Delta W = \nabla f(\underset{\sim}{x}_0) \cdot (\underset{\sim}{x} - \underset{\sim}{x}_0) + o(1)(\underset{\sim}{x} - \underset{\sim}{x}_0)$$

$o(1) \longrightarrow 0$ as $\underset{\sim}{x} \longrightarrow \underset{\sim}{x}_0$.