Recall: \( w = f(x, y, z) \)

Think: \( w = \text{temperature} \Theta (x, y, z) \)

We defined:

\[
\nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = \nabla f(x, y, z)
\]

\[
f_x = \frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y, z) - f(x, y, z)}{\Delta x}
\]

"Diff wrt \( x \) holding \( y \) and \( z \) fixed as const".

\[
f(x, y, z) = x^2 y - yz
\]

\[
\frac{\partial f}{\partial x} = zxy \quad \frac{\partial f}{\partial y} = x^2 - z \quad \frac{\partial f}{\partial z} = -y
\]

\[
\nabla f(x, y, z) = (zxy, x^2 - z, -y)
\]
The importance of $\mathbf{\nabla} f$ from Chain Rule

$$\frac{d}{dt} f(\mathbf{r}(t)) = \mathbf{\nabla} f \cdot \mathbf{r}'(t)$$

$$\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{r}_0 + t \mathbf{v}$$

$$\mathbf{r}'(t) = \mathbf{v}$$

$$|\mathbf{r}'(t)| = \frac{ds}{dt} = |\mathbf{v}|$$

**Conclude:** $\mathbf{v}$ unit $\Rightarrow$ $ds = dt$

$$\therefore \frac{ds}{dV} = \mathbf{\nabla} f \cdot \frac{\mathbf{v}}{|\mathbf{v}|}$$

"rate of change of temp unit arc length in direction $\mathbf{v}$ from $\mathbf{r}_0$"
Since \( \frac{df}{dx} = \lim_{\Delta x \to 0} \)

we need to talk about limits.

**Defn:** We say \( f \) is **continuous** at \( x_0 = (x_0, y_0, z_0) \) if \( \lim_{x \to x_0} f(x, y, z) = f(x_0, y_0, z_0) \)

Words: "The graph of \( f \) is unbroken at \( x_0 \)."

We need to define limits in \( \mathbb{R}^3 \).
**Definition:** \( \lim_{x \to x_0} f(x) = L \) if

\[ x \to x_0 \]

Words: "The outputs of \( f \) get arbitrarily close to \( L \) if the inputs \( x \) get sufficiently close to \( x_0 \)."

Mathematically: \( \forall \varepsilon > 0 \ \exists \delta > 0 \) s.t.

if \( |x - x_0| < \delta \) then \( |f(x, \xi) - L| < \varepsilon \).

**Diagram**

- \( |x - x_0| < \delta \)
- \( W_0 \)
- \( |W - W_0| < \varepsilon \)
- Inputs \( x \)
- \( W = \text{output (temperature)} \)
Example: Prove: \( f(x,y,z) = xy^2 - z \) is continuous at \( x_0 = 0 = (0, 0, 0) \)

Note: \( f(0, 0, 0) = 0 \) so \( L = 0 \).

We must prove \( \lim_{x \to x_0} f(x) = L = 0 \).

Proof: To prove: \( \lim_{x \to 0} f(x) = 0 \)

Fix \( \varepsilon > 0 \). We find \( \delta > 0 \) such that

if \( |x| < \delta \) then \( |f(x) - 0| < \varepsilon \)

We find \( \delta \) s.t. if \( |x| < \delta \) then \( |xy^2 - z| < \varepsilon \)
Need: $|f(x)| < \varepsilon \iff |xy^2 - z| < \varepsilon$

Suffices to make $|xy^2| < \frac{\varepsilon}{2}$ and $|z| < \frac{\varepsilon}{2}$

Note: $|x| < |(x, y, z)| = \sqrt{x^2 + y^2 + z^2}$

$|\theta| < |x|$, $|z| < |x|$

So if $|x| < \frac{\varepsilon}{2}$ and $|y| < \frac{\varepsilon}{2}$ and $|z| < \frac{\varepsilon}{2}$

then $|xy^2| < \left|\frac{\varepsilon}{2}\left(\frac{\varepsilon}{2}\right)\right| < \varepsilon/2$, $|z| < \frac{\varepsilon}{2}$ ($\varepsilon > 0$)

So if sufficient to make $|z| < \frac{\varepsilon}{2}$

End of thinking

Choose $\delta = \frac{\varepsilon}{2}$. Then if $|x| < \frac{\varepsilon}{2}$

also $|x| < \frac{\varepsilon}{2}$, $|y| < \frac{\varepsilon}{2}$, $|z| < \frac{\varepsilon}{2}$

so $|xy^2 - z| < |x||y^2| + |z| < \frac{\varepsilon}{2}\left(\frac{\varepsilon}{2}\right) + \frac{\varepsilon}{2} < \varepsilon$
Example: Show that if
\[ w = f(x, y, z) = \frac{2xy + yz}{x^2 + y^2 + z^2} \quad z \neq 0 \]
then there is no way to define \( f \) at \( x = 0 \) so that \( f \) is cont.

Solut. To be cont., \( w \) has to have the same limit every way \( x \to 0 \).
Define \( x = \vec{r}(t) = t\, \vec{v} \quad \vec{v} = (a, b, c) \) arbitrary.

\[
\lim_{t \to 0} f(\vec{r}(t)) = \lim_{t \to 0} f(t\, \vec{v})
\]

\[
f(\vec{r}(t)) = f(t\, \vec{v}) = \frac{2x^2ab + x^2bc}{t^2(a^2 + b^2 + c^2)} = \frac{2ab + bc}{a^2 + b^2 + c^2}
\]

\[
\lim_{t \to 0} f(\vec{r}(t)) = \frac{2ab + bc}{a^2 + b^2 + c^2} \quad \text{depends on } \vec{v} \text{ not unique.}
\]
Theorem: \( \lim_{x \to x_0} f(x) = L \) if and only if for every sequence \( x_n \to x_0 \), we have \( \lim_{n \to \infty} f(x_n) = f(x_0) \).

Recall: \( \lim_{n \to \infty} x_n = x_0 \) if \( \forall \epsilon \in \mathbb{R} \), there exists \( N \in \mathbb{N} \) such that if \( n > N \) then \( |x_n - x_0| < \epsilon \).

Theorem: If \( f(x,y,z) \) is differentiable...
We'd like to say that if a function is differentiable at \((x, y, z)\) then it is continuous at \((x, y, z)\).

It's not enough to have \(f_x, f_y, f_z\) exist at \((x, y, z)\). For example:

\[ f(x, y) = \frac{xy}{x^2 + y^2} \quad f(0, 0) = 0 \]

Is not cont. at \((0, 0)\) but \(f_x, f_y, f_z\) do exist!

\[ \frac{\partial f}{\partial x} (0, 0) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, 0) - f(0, 0)}{\Delta x} = 0 \]

\[ \frac{\partial f}{\partial y} (0, 0) = 0 \]

Def: \(f\) diff \(\Rightarrow\) if \(f_x, f_y, f_z\) exist and

\[ \Delta w = f_x \Delta x + f_y \Delta y + f_z \Delta z + \]
\[ \Delta w = \nabla f(x_0) \cdot (x - x_0) + O(1) (x - x_0) \]

\[ O(1) \rightarrow 0 \text{ as } x \rightarrow x_0. \]