2) M.5/14.1 (i)

$\boxed{3}$ Differentiability $w = f(x,y,z)$ "temp @ position $\underset{\sim}{x} = (x,y,z)$"

- Recall: That $f_x = \frac{\partial f}{\partial x}$, f_y, f_z exist at point $\underset{\sim}{x}_0$ does not imply f is necessarily continuous at $\underset{\sim}{x}_0$.

f cont @ $\underset{\sim}{x}_0$ if $\lim\limits_{\underset{\sim}{x} \to \underset{\sim}{x}_0} f(\underset{\sim}{x}) = f(\underset{\sim}{x}_0)$

- Defn: f is differentiable at $\underset{\sim}{x}_0$ if

$$f(x,y,z) - f(x_0,y_0,z_0) = \nabla f_0 \cdot \overbrace{(x-x_0, y-y_0, z-z_0)}^{\underset{\sim}{x} - \underset{\sim}{x}_0}$$

$$+ o(1)|\underset{\sim}{x} - \underset{\sim}{x}_0|$$

"tends to zero as $\underset{\sim}{x} \longrightarrow \underset{\sim}{x}_0$"

It says – the linear correction $\nabla f_0 \cdot (\underset{\sim}{x} - \underset{\sim}{x}_0)$ becomes a better & better approx to changes in f ~~near~~ as $\underset{\sim}{x}$ gets closer to $\underset{\sim}{x}_0$.

2

- **Thm**: If f is diff @ $\underset{\sim}{x}_0$, then f is cont @ $\underset{\sim}{x}_0$.

(I.e., as $|\underset{\sim}{x}-\underset{\sim}{x}_0| \to 0$, $|w-w_0| \to 0$ ✓)

In text: f diff if

$$f(x,y,z)-f(x_0,y_0,z_0) = \nabla f_0 \cdot (\overbrace{x-x_0}^{\Delta x}, \overbrace{y-y_0}^{\Delta y}, \overbrace{z-z_0}^{\Delta z})$$

$$+ \underbrace{\varepsilon_1 \Delta x + \varepsilon_2 \Delta y + \varepsilon_3 \Delta z}_{\text{error}}$$

where $\varepsilon_i \to 0$ as $|\underset{\sim}{x}-\underset{\sim}{x}_0| \to 0$.

Same as saying error is $o(1)|\Delta \underset{\sim}{x}|$

$$|\Delta \underset{\sim}{x}| = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$$

Differentials: $dx = \Delta x$, $dy = \Delta y$, $dz = \Delta z$

but $\Delta w \approx dx$

$$\underbrace{f(x,y,z) - f(x_0,y_0,z_0)}_{\Delta w} = \nabla f_0 \cdot (\overset{dx}{x - x_0}, \overset{dy}{y - y_0}, \overset{dz}{z - z_0}) + o(1)|\underset{\sim}{x} - \underset{\sim}{x}_0|$$

$$\Delta w = \nabla f_1 \cdot (dx, dy, dz) + o(1)(\underset{\sim}{x} - \underset{\sim}{x}_0)$$

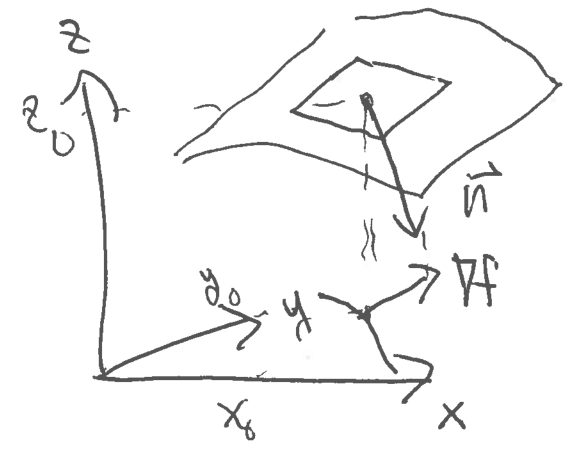
$$dw = \nabla f_0 \cdot (dx, dy, dz) + \cancel{\text{error}}$$

$$= \underbrace{f_x\, dx + f_y\, dy + f_z\, dz}_{\text{linear approx to } \Delta w}$$

"dw is the approx ~~[illegible]~~ change in w corresponding to a change along the <u>linear</u> <u>approx</u> to f at $\underset{\sim}{x}_0$"

② Case $n=2$: The linear approx is the tangent plane –

$\vec{n} = (f_x, f_y, -1)$



$z = f(x,y)$

$$z - z_0 = \nabla f \cdot (x-x_0, y-y_0) + o(1)\,|\underline{x} - \underline{x}_0|$$

Ignoring the error,

$$z - z_0 = (f_x, f_y)(x-x_0, y-y_0)$$

equn for a plane:

$$\underbrace{(f_x)}_{a} x + \underbrace{(f_y)}_{b} y \underbrace{-}_{c} z = \underbrace{(f_x)x_0 + (f_y)y_0 - z_0}_{d}$$

$$\vec{n} = \overrightarrow{(f_x, f_y, -1)}$$

<u>Theorem</u>: If ~~the~~ the partial derivatives of f exist & are continuous in an open set, then f is differentiable at every pt. (Skip Pf)

② Comment on Chain Rule —

$f(x,y,z)$ $\qquad$ $\vec{r}(t) = \overrightarrow{(x(t), y(t), z(t))}$

$$\frac{d}{dt} f(\vec{r}(t)) = \nabla f \cdot \vec{r}\,'(t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

What if $\vec{r} = \vec{r}(s,t) \overset{?}{=} x(r,s)\underset{\sim}{i} + y(r,s)\underset{\sim}{j} + z(r,s)\underset{\sim}{k}$

$$\frac{\partial}{\partial s} f(\vec{r}(s,t)) = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$
$$= \nabla f \cdot \vec{r}_s$$

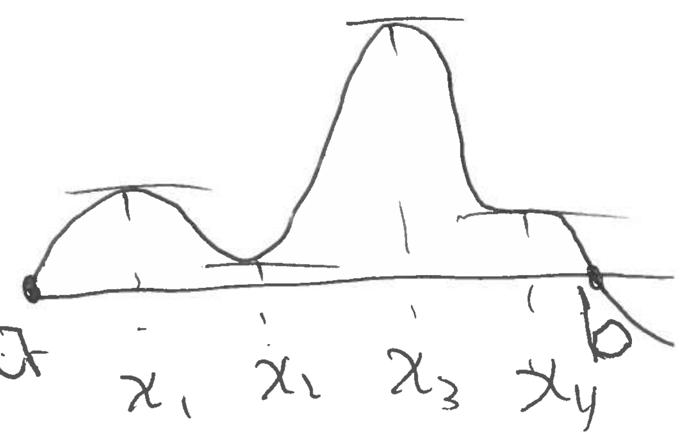
$$\frac{\partial}{\partial t} f(\vec{r}(s,t)) = \nabla f \cdot \vec{r}_t$$

Next Topic: Max/Min Problems for

$$\boxed{z = f(x,y)} \quad f: \mathbb{R}^2 \to \mathbb{R}$$
$$(x,y) \longmapsto z$$

- Recall 1-d $y=f(x)$

To find max/min



① Solve $f'(x)=0$

x_0 a crit pt if $f'(x_0)=0$

② Check 2nd deriv: $f''(x_0)$

x_0 crit and $f''(x_0)<0 \iff$ rel min

$f''(x_0)>0 \iff$ rel max

$f''(x_0)=0 \iff$ inflection pt

Thm: Assume f diff on $[a,b]$. Then f takes its max value at –

(1) A rel max (2) boundary $x=a$, $x=b$ or (3) at an inflection pt.

Corresponding Theory for $z=f(x,y)$

① Solve $\nabla f(\underset{\sim}{x})=0$

$\underset{\sim}{x}_0$ is a Critical pt if $\nabla f(\underset{\sim}{x}_0)=0$

② Check 2nd Derivative

$$H=\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}(\underset{\sim}{x}_0) \quad |H| = f_{xx}f_{yy}-f_{xy}f_{yx} = \operatorname{Det} H(\underset{\sim}{x}_0)$$

$f_{xx}<0$, $\operatorname{Det} H>0 \Leftrightarrow$ Rel Min

$f_{xx}>0$, $\operatorname{Det} H>0 \Leftrightarrow$ Rel Max

$\operatorname{Det} H=0 \Leftrightarrow$ inflection

<u>Thm</u>: If f is diff on a <u>closed</u> rectangle $a \le x \le b$, $c \le y \le d$. Then f takes its max value at

1. A rel Max
2. On boundary $x = a, x = b, y = a, y = b$

<u>or</u>

3. At an inflection point.

If f not diff @ points, then max/min could be taken on there !