$\boxed{\text{Lagrange multiplers}}$

MAT 21C W 2020
Temple

- Consider $w = f(x, y, z)$
  Again, think of w as the temperature at point $(x, y, z) \in \mathbb{R}^3$

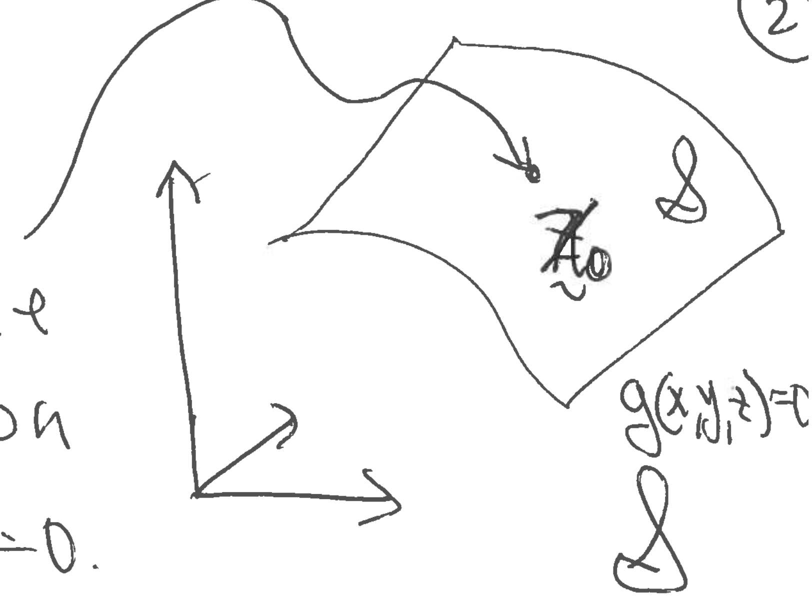
- We ask this problem —
  Maximize f on the level surface of a function $g(x, y, z)$; say

$$g(x, y, z) = 0$$

(Note: if $g(x, y, z) = 2$ is the level surface, then $g(x, y, z) - 2 = 0$ & we reduce to the zero level surface by simply replacing g by $g-2$.)

<u>Picture</u>:

Maximize $f(x,y,z)$ on $g(x,y,z)=0$.



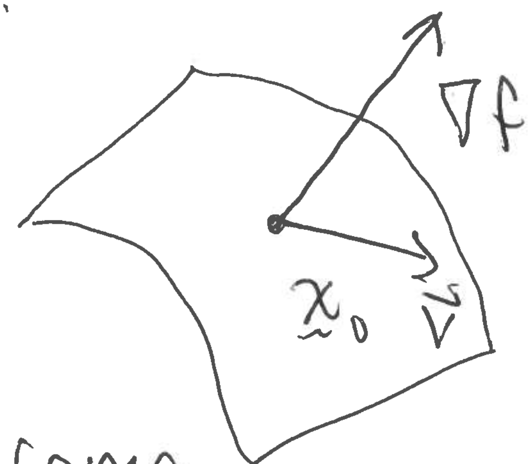
• Now assume the max temperature w occurs at $\underset{\sim}{x}_0$ on surface, so

$$g(\underset{\sim}{x}_0)=0.$$

Ie, $w_0=f(x_0,y_0,z_0)$ is the max temp on surface $g(\underset{\sim}{x})=0$.

<u>Claim</u>: $\nabla f(x_0,y_0,z_0)$ must point <u>normal</u> to surface at the max!

<u>Claim</u>: $\nabla f(\underset{\sim}{x}_0) \perp \mathcal{S}$.



To see this - assume ∇f not $\perp$ to $\mathcal{S}$.

Then $\nabla f \cdot \vec{v} \neq 0$ for some vector tangent to surface -

If $\nabla f \cdot \vec{v} < 0$ then $\nabla f(-\vec{v}) > 0$ so replace $\vec{v}$ with $-\vec{v}$, and we can assume $\nabla f \cdot \vec{v} > 0$.

But: $\frac{d}{dt} f(\underset{\sim}{x}_0 + t\vec{v}) = \nabla f \cdot \vec{v} > 0$ implies temperature <u>increases</u> moving in direction $\vec{v}$ from $\underset{\sim}{x}_0$ in $\mathcal{S}$.

But then w increases from w_0 moving in surface S away from $\underline{x}_0 \Rightarrow w_0$ not the max temp in S!

Conclude: At the max temp in S, $\nabla f \perp S$.

- Now we know $\nabla g \perp S$ also because the gradient points normal to the level surface: thus we have:

<u>Theorem</u>: At a local max/min of $\theta = f(x,y,z)$ on the level surface $g(x,y,z) = \text{const}$, we must have

$$\nabla f = \lambda \nabla g$$

for some scalar λ.

<u>Defn</u>: The number λ is called a <u>Lagrange Multiplier</u>

Here is how we use this principle:

<u>Problem</u>: Maximize (or minimize) $w = f(x,y,z)$ subject to the constraint $g(x,y,z) = 0$.

We know @ such a point:

$$\boxed{\begin{array}{c} \nabla f = \lambda \nabla g \\ g = 0 \end{array}}$$

Now build new function:

$$F(x,y,z,\lambda) = f - \lambda g$$

<u>Claim</u>: The condition that

$$\nabla F = 0$$

is exactly what we want:

Ie. $\nabla F = (F_x, F_y, F_z, F_\lambda)$

$$\left.\begin{aligned} F_x &= f_x - \lambda g_x \\ F_y &= f_y - \lambda g_y \\ F_z &= f_z - \lambda g_z \end{aligned}\right\} \nabla f - \lambda g = 0 \checkmark$$

$$F_\lambda = -g \qquad \Rightarrow g = 0 \checkmark$$

<u>Conclude</u>: By including the extra variable λ into $f - \lambda g = F$ we get that $\nabla F = 0$ is the condt that

w is a critical point ~~of~~ the temperature, on the level surface $g = 0$.

<u>Example</u>: Maximize

$$f(x,y,z) = 2x + y - z$$

on the sphere of radius $R = \frac{1}{2}$

$$x^2 + y^2 + z^2 = \frac{1}{4}$$

<u>Soln</u>: $F(x,y,z,\lambda) = f - \lambda g$

$$= 2x + y - z - \lambda\left(x^2 + y^2 + z^2 - \frac{1}{4}\right)$$

$$\nabla F = 0 \iff \nabla f - \lambda g = 0 \overset{①}{\iff} \overrightarrow{(2,1,-1)} - 2\lambda\overrightarrow{(x,y,z)} = 0$$

$$F_\lambda = -g = 0 \overset{②}{\iff} x^2 + y^2 + z^2 - \frac{1}{4} = 0$$

1 $\overrightarrow{(2,1,-1)} - 2\lambda(x,y,z) = 0$

$\Leftrightarrow \quad x = \frac{1}{\lambda}, \quad y = \frac{1}{2\lambda}, \quad z = -\frac{1}{2\lambda}$

2 $x^2+y^2+z^2-\frac{1}{4} = 0$

$\Leftrightarrow \quad \frac{1}{\lambda^2} + \frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} - \frac{1}{4} = 0$

$\Leftrightarrow \quad 1 + \frac{1}{4} + \frac{1}{4} - \frac{1}{4}\lambda^2 = 0$

$\Leftrightarrow \quad \lambda^2 = 4+1+1-1 = 5 \quad \boxed{\lambda = \pm\sqrt{5}}$

$\therefore \quad x_\pm = \frac{1}{\pm\sqrt{5}} \quad y = \frac{1}{\pm 2\sqrt{5}} \quad z = -\frac{1}{\pm 2\sqrt{5}}$

are the two critical points.

$\underset{\sim}{x}_- = \frac{1}{\sqrt{5}}\left(-1, -\frac{1}{2}, \frac{1}{2}\right) \quad \underset{\sim}{x}_+ = \frac{1}{\sqrt{5}}\left(1, \frac{1}{2}, -\frac{1}{2}\right)$

Plug in to see which one is the rel max: $f(\underset{\sim}{x}) = 2x + y - z$

$$f(\underset{\sim}{x}_-) = \frac{1}{\sqrt{5}}\left(-2 - \frac{1}{2} - \frac{1}{2}\right) = -\frac{3}{\sqrt{5}}$$

$$f(\underset{\sim}{x}_+) = \frac{1}{\sqrt{5}}\left(2 + \frac{1}{2} \cdot \frac{1}{2}\right) = \frac{3}{\sqrt{5}}$$

Max Temp on shore is $\frac{3}{\sqrt{5}}$!

<u>Summary</u>: Since we know $\nabla f = \lambda \nabla g$ at the critical pt, we can solve for all the (x,y,z) where $\nabla f = \lambda \nabla g$ to get (x,y,z) as a fn of λ ! Then plug this into $g(x,y,z) = 0$ & we get one equation for λ, which can be solved — Great idea !