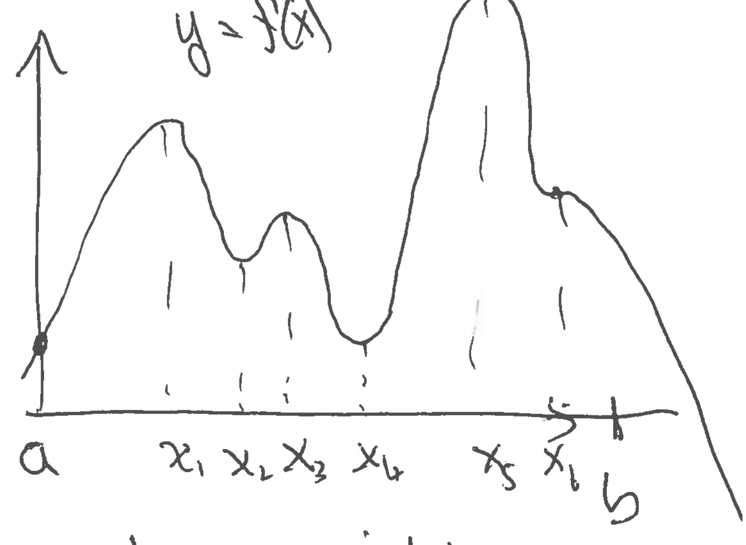
Max/Min Pts



- 1-D Theory

Problem: Find max/min of f on [a,b], f has two derivatives.

Thm: The max/min occurs at a critical pt where $f'(x)=0$ or at end pt $x=a$, $x=b$.

Defn: x_0 is a crit pt of f if $f'(x_0)=0$

x_0 is a rel max if $f''(x_0)<0$

x_0 is a rel min if $f''(x_0)>0$

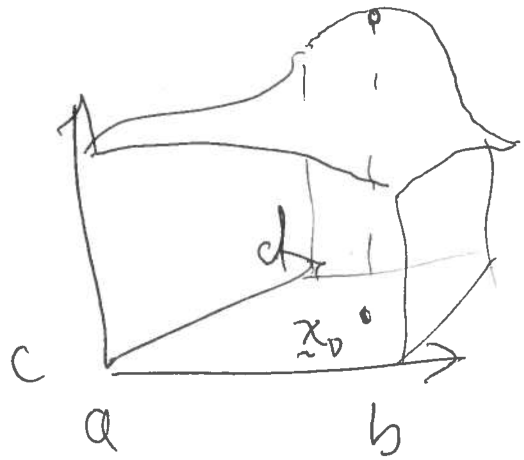
x_0 is an inflection pt if $f''(x_0)=0$

Note: If there are pts where $f, f'(x_0)$ does not exist, then these are also critical pts.

- How Theory Carries over to $\mathbb{R}^2$

Assume $z = f(x,y)$ has two derivatives

Find max/min of f on $[a,b] \times [c,d]$

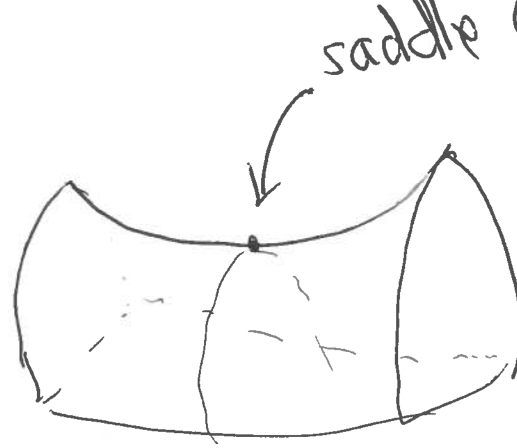


<u>Defn</u>: $\underset{\sim}{x}_0 = (x_0, y_0)$ is a <u>critical point</u> of f if

$\nabla f(\underset{\sim}{x}_0) = 0$.

Critical pts come in <u>four</u> types —

1. $\underset{\sim}{x}_0$ is a rel max $\left(|H| > 0,\ f_{xx} < 0\right)$
2. $\underset{\sim}{x}_0$ is a rel min $\left(|H| > 0,\ f_{xx} > 0\right)$
3. $\underset{\sim}{x}_0$ is a <u>saddle</u> pt $\left(|H| < 0\right)$
4. $\underset{\sim}{x}_0$ is an inflection pt $\left(|H| = 0\right)$



Saddle Point (new to $\mathbb{R}^2$)

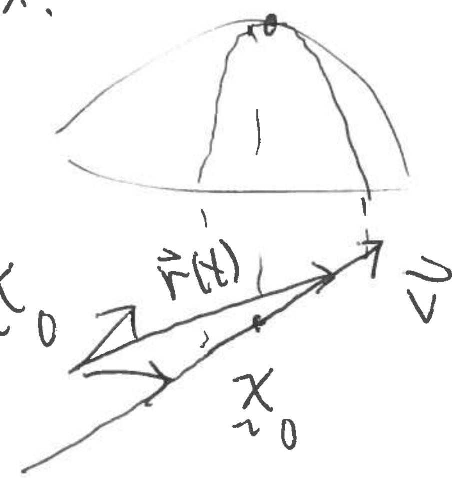
A saddle point is a local min in one direction & a local max in another direction.

Thm: max/min occurs at rel max/rel min/inflection or on boundary $x=a,b$ $y=c,d$ but never at a saddle pt.

How do we understand this —

Assume $\underset{\sim}{x}_0$ is a rel max.

Then its a rel max along every line thru $\underset{\sim}{x}_0$



• Line thru $\underset{\sim}{x}_0$:

$$\vec r(t) = \underset{\sim}{x}_0 + t\vec v, \quad -\infty < t < \infty$$

Consider $f(\vec r(t)) = f(\underset{\sim}{x}_0 + t\vec v)$

$$\frac{df}{dt} = \underbrace{\nabla f_0}_{0} \cdot \vec v = 0;$$ assume $\nabla f(\underset{\sim}{x}_0) = 0$

i.e. $\underset{\sim}{x}$ not critical

$$\frac{d^2 f}{dt^2} = \frac{d}{dt}\left(\frac{\partial f}{\partial x}(\vec r(t)), \frac{\partial f}{\partial y}(\vec r(t))\right) \cdot \vec V$$

$$= \left[\left(\nabla \frac{\partial f}{\partial x}\right) \cdot \vec r\,'(t) + \nabla \frac{\partial f}{\partial y} \cdot \vec r\,'(t)\right] \cdot \vec V$$

$$\frac{df}{dt} = \left(\frac{\partial f}{\partial x}(\underset{\sim}{x}_0 + t\vec{v}), \frac{\partial f}{\partial y}(\underset{\sim}{x}_0 + t\vec{v}) \right) \cdot \overrightarrow{(a, b)}$$

$$= \frac{\partial f}{\partial x}(\underset{\sim}{x}_0 + t\vec{v}) \cdot a + \frac{\partial f}{\partial y}(\underset{\sim}{x}_0 + t\vec{v}) \cdot b$$

$$\left.\frac{d^2 f}{dt^2}\right|_{t=0} = \left(\frac{\partial^2 f}{\partial x^2} a + \frac{\partial^2 f}{\partial y \partial x} \cdot b \right) a + \left(\frac{\partial^2 f}{\partial x \partial y} a + \frac{\partial^2 f}{\partial y^2} \cdot b \right) b$$

$$= (a, b) \underbrace{\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}}_{H \,\equiv\, \text{Hessian of } f \text{ @ } \underset{\sim}{x}_0} \begin{bmatrix} a \\ b \end{bmatrix}$$

$f_{xy} = \frac{\partial}{\partial y} \frac{\partial}{\partial x} f$

$\uparrow$ 1st $\uparrow$ 2nd

$f_{xy} = f_{yx}$

<u>Conclude</u>: For $\underset{\sim}{x}_0$ to be a rel min we need $\vec{v}^T H \vec{v} < 0 \quad \forall \; \vec{v} \neq 0$!

Similarly: For $\underset{\sim}{x}_0$ to be a rel min, we need $\vec{v}^T H \vec{v} > 0 \quad \forall \vec{v} \neq 0$.

Q: When will $\vec{v}^T H \vec{v}$ have the same sign for every $\vec{v} = \overrightarrow{(a,b)}$??

Ans: When $\boxed{\det H > 0}$

$$\boxed{f_{xx} f_{yy} - f_{xy}^2 > 0}$$

To determine whether its max/min check one value of $\vec{v}$, say $\vec{v} = (1,0)$.

Then $\quad (1,0) H \begin{pmatrix} 1 \\ 0 \end{pmatrix} = f_{xx} < 0 \quad$ rel max

$> 0 \quad$ rel min.

<u>Thm</u>: $\vec{v}^T H \vec{v}$ has one sign for every $\vec{v}$ iff $\det H > 0$.

"Proof": We need facts from matrix theory (linear alg)

$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ 2x2 matrix

(1) If $M\vec{R} = \lambda\vec{R}$ some vector $\vec{R}$ we say λ is an <u>eigenvalue</u> and $\vec{R}$ its eigenvector

(2) <u>Theorem</u> (Symmetric Matrix Thm <u>Big</u>)

A symmetric matrix has <u>real</u> eigenvalues and an orthonormal basis of eigenvectors. Ie $|\vec{R}_1| = 1 = |\vec{R}_2|$, $\lambda_1 \leq \lambda_2 \in \mathbb{R}$

$M\vec{R}_1 = \lambda_1 \vec{R}_1$, $MR_2 = \lambda_2 \vec{R}_2$, $\vec{R}_1 \cdot \vec{R}_2 = 0$

(3) $H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$ symmetric

$$\begin{bmatrix} - R_1 - \\ - R_2 - \end{bmatrix} H \begin{bmatrix} | & | \\ R_1 & R_2 \\ | & | \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

rows are eigenvectors

colms are e-vectors

$$\begin{bmatrix} - R_1 - \\ - R_2 - \end{bmatrix} \begin{bmatrix} | & | \\ R_1 & R_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(4) $|A \cdot B \cdot C| = |A||B||C|$ property of matrices

$$|A^{-1}| = \frac{1}{|A|}$$

$\therefore$

$$\cancel{\left|\begin{bmatrix} - R_1 - \\ - R_2 - \end{bmatrix}\right|} \, |H| \, \cancel{\begin{vmatrix} | & | \\ R_1 & R_2 \\ | & | \end{vmatrix}} = \lambda_1 \cdot \lambda_2$$

Conclude: $|H| > 0$ means both eigenvalues have the same sign

5 $R = \begin{bmatrix} | & | \\ R_1 & R_2 \\ | & | \end{bmatrix}$

$$R^T H R = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$H = R \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} R^T$$

$$\vec{v}^T H \vec{v} = \underbrace{\vec{v}^T R}_{\vec{b}^T} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \underbrace{R^T \vec{v}}_{\vec{b}} = \lambda_1 b_1^2 + \lambda_2 b_2^2$$

<u>same</u> sign $\forall \vec{v}$!

6 <u>Conclude</u>: $\frac{d^2}{dt^2} f(\underset{\sim}{x}_0 + t\vec{v})$ has <u>same</u> sign $\forall \vec{v}$ iff $\boxed{\operatorname{Det} H > 0}$!

Ex: A rectangular box of sides x, y, z has to satisfy $2x+2y+z=2$. Find box that max's Vol $V = xyz$.

Soln: $V = xyz = xy(2-2x-2y)$

$= 2xy(1-x-y)$

Crit pts. $\frac{\partial V}{\partial x} = \cancel{2}y(1-x-y) - \cancel{2}xy = 0$

$= 2y(1-2x-y) = -2y^2 + 2y - 4yx$

$\frac{\partial V}{\partial y} = \cancel{2}x(1-x-y) - \cancel{2}xy = 0$

$= 2x(1-x-2y) = -2x^2 + 2x - 4xy$

$1 - x - y - x = 0 \quad 1 - 2x - y = 0 \quad y = 1 - 2x$

$1 - x - y - y = 0 \quad 1 - x - 2y = 0$

$1 - x - 2(1-2x) = 0$

$\boxed{x = \frac{1}{3}, y = \frac{1}{3}}$

$$H = \begin{bmatrix} V_{xx} & V_{xy} \\ V_{yx} & V_{yy} \end{bmatrix} = \begin{bmatrix} -4y, & -4y-4x+2 \\ -4x-4y+2, & -4x \end{bmatrix}$$

$$H\left(\tfrac{1}{3},\tfrac{1}{3}\right) = \begin{bmatrix} -4/3, & -\frac{8}{3}+2 \\ -\frac{8}{3}+2 & -\frac{4}{3} \end{bmatrix}$$

$$= \begin{bmatrix} -4/3, & -\frac{6}{3} \\ -\frac{6}{3}, & -\frac{4}{3} \end{bmatrix}$$

$$\left|H\left(\tfrac{1}{3},\tfrac{1}{3}\right)\right| = \frac{16}{9} - \frac{12}{9} > 0 \checkmark$$

$$V_{xx} = -4/3 < 0 \Rightarrow \text{rel } \underline{\underline{\text{max}}} \checkmark$$

Note: $V=0$ on boundary $x=0$ or $y=0$

so only candidate is the crit pt!