Name: ______________________

Student ID#: ______________________

Section: ______________________

Final Exam

Thursday March 21, 10:30-12:30pm

MAT 21D, Temple, Winter 2019

Print name and ID's clearly. Have student ID ready. Write solutions clearly and legibly. Do not write near the edge of the paper or the stapled corner. Correct answers with no supporting work will not receive full credit. No calculators, notes, books, cellphones...allowed.

Problem	Your Score	Maximum Score
1		20
2		20
3		20
4		20
5		20
6		20
7		20
8		20
9		20
10		20
Total		200

Problem #1 (20pts): (a) Sketch the region of integration $\mathbf{R}_{xy}$ and evaluate the iterated integral

$$\int_0^1 \int_{1+(e-1)x}^{e^x} dydx. \tag{1}$$

Solution:

$$\int_{-1}^0 \int_{1+(e-1)x}^{e^x} dydx = \int_0^1 e^x - 1 - (e-1)x\,dx = e - 1 - 1 - \frac{e}{2} + \frac{1}{2} = \frac{e-3}{2}.$$

(b) Rewrite (1) with order of integration reversed. (Do not re-evaluate).

Solution:

$$\int_1^e \int_{\frac{y-1}{e-1}}^{ln(y)} dxdy.$$

Problem #2 (20pts): (a) Use polar coordinates to evaluate the integral

$$\int_0^\infty \int_0^\infty e^{-x^2-y^2}\, dxdy.$$

Solution:

$$\int_0^\infty \int_0^\infty = e^{-r^2} dxdy = \int_0^{\pi/2} \int_0^\infty = e^{-r^2} rdrd\theta$$

where we use $x = r\cos\theta$, $y = r\sin\theta$. Then using $u = -r^2$, $du = -2rdr$ we get

$$= -\frac{1}{2}\int_0^{\pi/2} e^{-r^2}|_0^\infty d\theta = \pi/4.$$

(b) Use part **(a)** to evaluate the Gaussian integral $\int_{-\infty}^{+\infty} e^{-x^2} dx$.

Solution:

$$\begin{aligned}\int_{-\infty}^\infty e^{-x^2} dx &= 2\int_0^\infty e^{-x^2} dx = \left(4\int_0^\infty e^{-x^2} dx \int_0^\infty e^{-y^2} dy\right)^{1/2} \\ &= \left(4\int_0^\infty \int_0^\infty e^{-x^2-y^2} dxdy\right)^{1/2} = \sqrt{\pi}\end{aligned}$$

Problem #3 (20pts): Assume

$$\vec{\mathbf{F}}(x,y,z) = (2xyz)\mathbf{i} + (x^2z + z^2)\mathbf{j} + (x^2y + 2yz - 2)\mathbf{k}.$$

(a) Find Div $\vec{\mathbf{F}}$.

Solution:

$$Div(\vec{\mathbf{F}}) = \overrightarrow{(2yz, 0, 2y)}$$

(b) Find Curl $\vec{\mathbf{F}}$.

Solution:

$$\begin{aligned} Curl(\vec{\mathbf{F}}) &= \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ 2xyz & x^2z + z^2 & x^2y + 2yz - 2 \end{pmatrix} \\ &= \mathbf{i}(x^2 + 2z - x^2 - 2z) - \mathbf{j}(2xy - 2xy) + \mathbf{k}(2xz - 2xz) = 0. \end{aligned}$$

(c) Use the method of partial integration to find an f such that $\mathbf{F} = \nabla f$.

Solution:

$$\begin{aligned}
\frac{\partial f}{\partial x} &= 2xyz, \quad f = x^2yz + g(y,z) \\
\frac{\partial f}{\partial y} &= x^2z + \frac{\partial g}{\partial y} = x^2z + z^2, \quad \frac{\partial g}{\partial y} = z^2, \quad g = z^2y + h(z) \\
\frac{\partial f}{\partial z} &= x^2y + 2zy + h'(z) = x^2y + 2yz - 2, \quad h'(z) = -2 \quad h = -2z
\end{aligned}$$

thus

$$f(x,y,z) = x^2yz + z^2y - 2z.$$

(d) Evaluate $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$ along any smooth curve C taking $A = (1,-1,2)$ to $B = (-1,1,1)$.

Solution:

$$\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}} ds = f(1,-1,1) - f(1,-2,2) = (1+1-2) - (-4-8-4) = 16.$$

Problem #4 (20pts): Let $\vec{\mathbf{v}} \equiv \vec{\mathbf{F}} = x^2\mathbf{i} + xy\mathbf{j} + z\mathbf{k}$ be the velocity field of a moving fluid.

(a) Find the unit vector in the direction of the axis of maximal circulation per area at point $P = (1, 2, 0)$.

Solution:

$$Curl(\vec{\mathbf{v}}) = \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ x^2 & xy & z \end{pmatrix} = \mathbf{i}(0-0) - \mathbf{j}(0-0) + \mathbf{k}(y) = \overrightarrow{(0, 0, y)}.$$

Ans: $\vec{\mathbf{n}} = \overrightarrow{(0, 0, 1)} = \mathbf{k}$.

(b) Find the maximal circulation per area at point $P = (1, 2, 0)$.

Solution:

Ans: $\|Curl(\vec{\mathbf{v}})(1, 2, 0)\| = 2$.

(c) Find the circulation per area around axis $\vec{\mathbf{w}} = \overrightarrow{(1,1,1)}$ at point $P = (1,2,0)$.

Solution:

Ans: $Curl(\vec{\mathbf{v}}) \cdot \frac{\vec{\mathbf{w}}}{\|\vec{\mathbf{w}}\|}|_{(1,2,0)} = \frac{2}{\sqrt{3}}$.

(d) Describe all axes $\vec{\mathbf{n}}$ around which there is zero circulation per area at point $P = (1,2,0)$.

Solution:

Ans: $Curl(\vec{\mathbf{v}}) \cdot \vec{\mathbf{n}} = 0$ so $\vec{\mathbf{n}}$ in the xy-plane.

Problem #5 (20pts): (a) Let $F = M\mathbf{i}+N\mathbf{j}+P\mathbf{k}$, let C be a smooth curve that takes A to B, and let $\vec{\mathbf{r}}(t)$ be a parameterization of C. Use Leibniz's substitution principle to show the following are equal:

$$\int_C \vec{\mathbf{F}}\cdot\vec{\mathbf{T}}\ ds = \int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} = \int_C \vec{\mathbf{F}}\cdot\vec{\mathbf{v}}\ dt = \int_C Mdx + Ndy + Pdz.$$

Solution: Let $\vec{\mathbf{r}}(t)$, $t_A \le t \le t_B$ be a parameterization of $\mathcal{C}$. Then

$$\vec{\mathbf{r}}'(t) = \frac{d\vec{\mathbf{r}}(t)}{dt} = \vec{\mathbf{v}} = \frac{ds}{dt}\vec{\mathbf{T}},$$

so

$$d\vec{\mathbf{r}} = \vec{\mathbf{v}}dt = \vec{\mathbf{T}}ds.$$

Also

$$d\vec{\mathbf{r}} = \overrightarrow{(dx, dy, dz)},\quad \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} = Mdx + Ndy + Pdz.$$

Substituting these Leibniz differential relations into the above integrals takes the first to the last.

Problem #6 (20pts): Recall the Divergence Theorem:

$$\int\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{n}\,d\sigma = \int\int\int_{\mathcal{V}} Div(\vec{\mathbf{F}})dV.$$

(a) Let $a \in \mathcal{R}$. Find a vector field $\vec{\mathbf{F}} = M\vec{\mathbf{i}} + N\vec{\mathbf{j}} + P\vec{\mathbf{k}}$ such that the flux of $\vec{\mathbf{F}}$ through the boundary of any volume $\mathcal{V}$ is always equal to the a times the volume itself.

Solution: We need $Div(\vec{\mathbf{F}}) = M_x + N_y + P_z = a$. So eg choose

$$M = \frac{ax}{3}, \quad N = \frac{ay}{3}, \quad P = \frac{az}{3}.$$

Thus the Divergence Theorem gives

$$\int\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{n}\,d\sigma = \int\int\int_{\mathcal{V}} Div(\vec{\mathbf{F}})dV = aVol(\mathcal{V}).$$

(b) Assume $\vec{\mathbf{F}} = Curl(\vec{\mathbf{G}})$ for some vector field $\vec{\mathbf{G}}$. Find all possible values of a in this case.

Solution: Since $Div(Curl(\vec{\mathbf{G}})) = 0$, $\int\int\int_{\mathcal{V}} Div(\vec{\mathbf{F}})dV = 0$, so $a = 0$.

(c) Now assume $\vec{\mathbf{F}} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is the mass flux vector $\delta\vec{\mathbf{v}}$ associated with a density δ being transported by a velocity $\mathbf{v}$. Find the rate at which mass is passing outward through the surface of the volume obtained by removing the cone $\phi \le \pi/4$ from the sphere $x^2+y^2+z^2 = 9$. (Hint: Spherical Coordinates.)

Solution: The mass per time passing outward through $\mathcal{S}$ is the flux

$$\int\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{n}\,d\sigma = \int\int\int_{\mathcal{V}} Div(\vec{\mathbf{F}})dV = \int\int\int_{\mathcal{V}} 3\,dV.$$

But

$$\begin{aligned}
\int\int\int_{\mathcal{V}} dV &= \int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^3 \rho^2\sin(\phi)d\rho d\phi d\theta \\
&= \int_0^{2\pi}\int_{\pi/4}^{\pi/2} \frac{\rho^3}{3}|_0^3\ \sin(\phi)d\phi d\theta \\
&= 9\cdot 2\pi\int_{\pi/4}^{\pi/2}\sin(\phi)d\phi = 18\pi(\cos(\phi))\ |_{\pi/2}^{\pi/4} \\
&= 18\pi[-\cos(\pi/2)+\cos(\pi/4)] = 9\sqrt{2}\pi.
\end{aligned}$$

Problem #7 (20pts): Recall that the volume of a sphere of radius R is $V = \frac{4}{3}\pi R^3$. Use this, together with the change of variables $x = au$, $y = bv$, $z = cw$ to derive the volume inside the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = R^2.$$

Solution: Let $u = \frac{x}{a},\ v = \frac{y}{a},\ w = \frac{z}{a}$. The the ellipsoid $\mathcal{E}_{xyz}$ in $xyz - space$ goes over to the sphere $u^2 + v^2 + w^2 = R^2$ of radius R in uvw-space, call it $\mathcal{E}_{uvw}$. Thus

$$\int\int\int_{\mathcal{E}_{xyz}} dxdydz = \int\int\int_{\mathcal{E}_{uvw}} \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw,$$

where

$$|J| \equiv \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = |diag(a,b,c)| = abc.$$

Thus

$$\int\int\int_{\mathcal{E}_{xyz}} dxdydz = abc\int\int\int_{\mathcal{E}_{uvw}} dudvdw = abc\cdot\frac{4}{3}\pi R^3.$$

Problem #8 (20pts): Recall the chain rule for functions of three variables:

$$\frac{d}{dt}f(x(t), y(t)) = f_x\dot{x} + f_y\dot{y} + f_z\dot{z}.$$

(**a**) Use the chain rule to prove that if a vector field $\vec{\mathbf{F}} = (M, \vec{N}, P)$ is conservative, (i.e. $\vec{\mathbf{F}}(x, y, z) = \nabla f(x, y, z)$ for some scalar function f), then

$$\int_C \vec{F} \cdot \vec{\mathbf{T}} ds = f(B) - f(A),$$

for any smooth curve C in $\mathcal{R}^3$ taking A to B.

Solution: Let $\vec{\mathbf{r}}(t)$ be a parameterization of C, $t_A \leq t \leq t_B$, $\vec{\mathbf{r}}(t_A) = A$, $\vec{\mathbf{r}}(t_B) = B$.

$$\begin{aligned}
\int_C \vec{F} \cdot \vec{\mathbf{T}} ds &= \int_{t_A}^{t_B} \vec{\nabla} f \cdot \vec{\mathbf{v}} dt \\
&= \int_{t_A}^{t_B} \frac{d}{dt} f(\vec{\mathbf{r}}(t)) dt \\
&= f(\vec{\mathbf{r}}(t_B)) - f(\vec{\mathbf{r}}(t_A)) = f(B) - f(A).
\end{aligned}$$

(**b**) Use the product rule for the dot product to prove that if $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$, where $\vec{\mathbf{a}} = \mathbf{r}''(t)$ when $\mathbf{r}(t)$ is the parametrization with respect to time, then

$$\int_C \vec{F} \cdot \vec{\mathbf{T}} ds = \frac{1}{2} m v_B^2 - \frac{1}{2} m v_A^2.$$

Solution:

$$\begin{aligned}\int_C \vec{F} \cdot \vec{\mathbf{T}} ds &= \int_{t_A}^{t_B} \vec{F} \cdot \vec{\mathbf{v}} dt \\ &= \int_{t_A}^{t_B} \vec{a} \cdot \vec{\mathbf{v}} dt \\ &= \int_{t_A}^{t_B} \frac{d}{dt} \frac{1}{2} \vec{\mathbf{v}} \cdot \vec{\mathbf{v}} dt \\ &= \frac{1}{2} \|\vec{\mathbf{v_B}}\|^2 - \frac{1}{2} \|\vec{\mathbf{v_A}}\|^2\end{aligned}$$

(c) Assume further that $\mathbf{F} = m\mathbf{a}$, and $\vec{\mathbf{F}}$ is conservative, so $\vec{\mathbf{F}} = -\nabla P$. Derive the principle of conservation of energy

$$\left\{\frac{1}{2} m v_B^2 - \frac{1}{2} m v_A^2\right\} + \{P(B) - P(A)\} = 0. \tag{2}$$

(Hint: Integrate $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}} \, ds$ two different ways.)

Solution: By part (**b**) $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}} \, ds = \left\{\frac{1}{2} m v_B^2 - \frac{1}{2} m v_A^2\right\}$. By part (**a**) $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}} \, ds = f(B) - f(A)$. Thus putting these together and using $f = -P$ gives

$$\left\{\frac{1}{2} m v_B^2 - \frac{1}{2} m v_A^2\right\} + \{P(B) - P(A)\} = 0.$$

Problem #9 (20pts): Let $\vec{\mathbf{F}} = \frac{-y}{r^2}\mathbf{i} + \frac{x}{r^2}\mathbf{j}$ where $r^2 = x^2 + y^2$.

(a) Find $Curl(\vec{\mathbf{F}})$.

Solution:

$$Curl(\vec{\mathbf{F}}) = \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ -\frac{y}{r^2} & \frac{x}{r^2} & 0 \end{pmatrix} = \mathbf{i}(0) - \mathbf{j}(0) + \mathbf{k}\left(\left(\frac{x}{r^2}\right)_x + \left(\frac{y}{r^2}\right)_y\right).$$

But

$$\begin{aligned}
\left(\frac{x}{r^2}\right)_x &= \frac{1}{r^2} - 2\frac{x}{r^3}\frac{x}{r} = \frac{1}{r^2} - 2\frac{x^2}{r^4} \\
\left(\frac{y}{r^2}\right)_y &= \frac{1}{r^2} - 2\frac{y}{r^3}\frac{y}{r} = \frac{1}{r^2} - 2\frac{y^2}{r^4} \\
Curl(\vec{\mathbf{F}}) &= \mathbf{k}\left(\frac{1}{r^2} - 2\frac{x^2}{r^4} + \frac{1}{r^2} - 2\frac{y^2}{r^4}\right) = 0.
\end{aligned}$$

(b) Define what a simply connected region is, and use this to explain why the following integral is the same for every positively oriented simple closed curve $\mathcal{C}$ surrounding the origin $(x, y) = (0, 0)$.

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T} ds = ? \tag{3}$$

Solution: A set $\mathcal{D}$ is simply connected if all loop can be contracted to a point without passing out of $\mathcal{D}$. [Draw two s.c.c's $\mathcal{C}_1$ and $\mathcal{C}_2$ around origin with connection curves $\pm\Gamma$, and argue that $\mathcal{C}_1 + \Gamma - \mathcal{C}_2 - \Gamma$ is a simple closed curve around a simply connected domain where $Curl(\vec{\mathbf{F}}) = 0$, so]

$$\int_{\mathcal{C}_1 + \Gamma - \mathcal{C}_2 - \Gamma} \vec{\mathbf{F}} \cdot \vec{\mathbf{T}} ds = 0 = \int_{\mathcal{C}_1} \vec{\mathbf{F}} \cdot \vec{\mathbf{T}} ds - \int_{\mathcal{C}_2} \vec{\mathbf{F}} \cdot \vec{\mathbf{T}} ds.$$

(c) Evaluate (3) by direct parameterization taking $\mathcal{C}$ to be the unit circle.

Solution: Let $\vec{\mathbf{r}}(t) = \overrightarrow{(\cos t, \sin t)}$ parameterize the unit circle $\mathcal{C}_0$, so

$$\vec{\mathbf{r}}'(t) = \overrightarrow{(-\sin t, \cos t)}.$$

Then

$$\int_{\mathcal{C}_0} \vec{\mathbf{F}} \cdot \vec{\mathbf{T}} ds = \int_0^{2\pi} \overrightarrow{(-\sin t, \cos t)} \cdot \overrightarrow{(-\sin t, \cos t)} dt = 2\pi$$

(d) Green's Theorem says $\int_{\mathcal{C}} \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds = \int\int_{\mathcal{A}} Curl(\vec{\mathbf{F}}) \cdot \mathbf{k}\, dA$. Explain why Green's Theorem fails in the case of (3).

Solution: $Curl(\vec{\mathbf{F}}) \neq 0$ in a simply connecte region due to singularity in $\vec{\mathbf{F}}$ at $r = 0$. In other words, $\vec{\mathbf{F}}$ itself cannot be extended to the singularity $r = 0$ even though $Curl(\vec{\mathbf{F}})$ can.

Problem #10 (20pts): (a) Consider the vector field $\vec{\mathbf{F}} = -\frac{\vec{\mathbf{r}}}{r^3}$ where $r = \sqrt{x^2+y^2+z^2}$. (This is Newton's inverse square force field with all constants set equal to one.) We know $\nabla\frac{1}{r} = \vec{\mathbf{F}}$, so $Curl(\vec{\mathbf{F}}) = 0$.

(a) Use $\frac{\partial}{\partial x}r = \frac{x}{r}$, etc, to calculate $Div(\vec{\mathbf{F}})$.

Solution: $\vec{\mathbf{F}} = -\overrightarrow{(\frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3})}$. Thus

$$Div(\vec{\mathbf{F}}) = -\left(\frac{x}{r}\right)_x - \left(\frac{y}{r}\right)_y - \left(\frac{z}{r}\right)_z,$$

$$\left(\frac{x}{r}\right)_x = \frac{\partial}{\partial x}\left(\frac{x}{r}\right) = \frac{1}{r^3} - 3\frac{x}{r^4}\frac{x}{r},$$

and similarly for $_y$ and $_z$, so

$$-Div(\vec{\mathbf{F}}) = \frac{3}{r^3} - 3\frac{x^2+y^2+z^2}{r^5} = 0.$$

(b) Calculate the flux $\int\int_{\mathcal{S}_R} \vec{\mathbf{F}} \cdot \mathbf{n}\, d\sigma$ where $\mathcal{S}_R$ is a sphere of radius R, (i.e., the surface of a ball of radius R.) You may use that the area of a sphere is $4\pi R^2$.

Solution:

$$\begin{aligned}\int\int_{\mathcal{S}_R} \vec{\mathbf{F}} \cdot \vec{\mathbf{n}} d\sigma &= \int\int_{\mathcal{S}_R} -\frac{\vec{\mathbf{r}}}{R^3} \cdot \frac{\vec{\mathbf{r}}}{R} d\sigma \\ &= -\frac{1}{R^2}\int\int_{\mathcal{S}_R} d\sigma = -\frac{1}{R^2} \cdot 4\pi R^2 = -4\pi.\end{aligned}$$

(c) Write the Divergence Theorem for $\vec{\mathbf{F}}$ and $\mathcal{S}_{\mathcal{R}}$ and explain why it fails for $\vec{\mathbf{F}} = -\frac{\vec{\mathbf{r}}}{r^3}$.

Solution: $\int\int_{\mathcal{S}_R} \vec{\mathbf{F}} \cdot \vec{\mathbf{n}} d\sigma = \int\int\int_{\mathcal{V}} Div(\mathbf{F}) dV$ and the LHS equals -4π while the RHS appears to equal zero because $Div(\mathbf{F}) = 0$.
The problem is that $Div(\vec{\mathbf{F}}) \neq 0$ at the singularity $r = 0$, so the RHS 3-d Riemann integral is not known to exist, and therefore cannot correctly be set equal zero.