Name: ______________________

Student ID#: ______________________

Section: ______________________

Final Exam

Thursday March 21, 10:30-12:30pm

MAT 21D, Temple, Winter 2019

Print name and ID's clearly. Have student ID ready. Write solutions clearly and legibly. Do not write near the edge of the paper or the stapled corner. Correct answers with no supporting work will not receive full credit. No calculators, notes, books, cellphones...allowed.

Problem	Your Score	Maximum Score
1		20
2		20
3		20
4		20
5		20
6		20
7		20
8		20
9		20
10		20
Total		200

Problem #1 (20pts): (a) Sketch the region of integration $\mathbf{R}_{xy}$ and evaluate the iterated integral

$$\int_0^1 \int_{1+(e-1)x}^{e^x} dy dx. \tag{1}$$

(b) Rewrite (1) with order of integration reversed. (Do not re-evaluate).

Problem #2 (20pts): (a) Use polar coordinates to evaluate the integral

$$\int_0^\infty \int_0^\infty e^{-x^2-y^2}\, dxdy.$$

(b) Use part **(a)** to evaluate the Gaussian integral $\int_{-\infty}^{+\infty} e^{-x^2} dx$.

Problem #3 (20pts): Assume

$$\vec{\mathbf{F}}(x, y, z) = (2xyz)\mathbf{i} + (x^2z + z^2)\mathbf{j} + (x^2y + 2yz - 2)\mathbf{k}.$$

(a) Find Div $\vec{\mathbf{F}}$.

(b) Find Curl $\vec{\mathbf{F}}$.

(c) Use the method of partial integration to find an f such that $\mathbf{F} = \nabla f$.

(d) Evaluate $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$ along any smooth curve C taking $A = (1, -1, 2)$ to $B = (-1, 1, 1)$.

Problem #4 (20pts): Let $\vec{\mathbf{v}} \equiv \vec{\mathbf{F}} = x^2\mathbf{i} + xy\mathbf{j} + z\mathbf{k}$ be the velocity field of a moving fluid.

(a) Find the unit vector in the direction of the axis of maximal circulation per area at point $P = (1, 2, 0)$.

(b) Find the maximal circulation per area at point $P = (1, 2, 0)$.

(c) Find the circulation per area around axis $\vec{\mathbf{w}} = \overrightarrow{(1,1,1)}$ at point $P = (1,2,0)$.

(d) Describe all axes $\vec{\mathbf{n}}$ around which there is zero circulation per area at point $P = (1,2,0)$.

Problem #5 (20pts): (a) Let $F = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$, let C be a smooth curve that takes A to B, and let $\vec{\mathbf{r}}(t)$ be a parameterization of C. Use Leibniz's substitution principle to show the following are equal:

$$\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{v}}\, dt = \int_C M dx + N dy + P dz.$$

Problem #6 (20pts): Recall the Divergence Theorem:

$$\int\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{n}\,d\sigma = \int\int\int_{\mathcal{V}} Div(\vec{\mathbf{F}})dV.$$

(a) Let $a \in \mathcal{R}$. Find a vector field $\vec{\mathbf{F}} = M\vec{\mathbf{i}} + N\vec{\mathbf{j}} + P\vec{\mathbf{k}}$ such that the flux of $\vec{\mathbf{F}}$ through the boundary of any volume $\mathcal{V}$ is always equal to the a times the volume itself.

(b) Assume $\vec{\mathbf{F}} = Curl(\vec{\mathbf{G}})$ for some vector field $\vec{\mathbf{G}}$. Find all possible values of a in this case.

(c) Now assume $\vec{\mathbf{F}} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is the mass flux vector $\delta\vec{\mathbf{v}}$ associated with a density δ being transported by a velocity $\mathbf{v}$. Find the rate at which mass is passing outward through the surface of the volume obtained by removing the cone $\phi \leq \pi/4$ from the sphere $x^2+y^2+z^2 = 9$. (Hint: Spherical Coordinates.)

Problem #7 (20pts): Recall that the volume of a sphere of radius R is $V = \frac{4}{3}\pi R^3$. Use this, together with the change of variables $x = au$, $y = bv$, $z = cw$ to derive the volume inside the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = R^2.$$

Problem #8 (20pts): Recall the chain rule for functions of three variables:

$$\frac{d}{dt}f(x(t),y(t)) = f_x\dot{x} + f_y\dot{y} + f_z\dot{z}.$$

(**a**) Use the chain rule to prove that if a vector field $\vec{\mathbf{F}} = (M, \vec{N}, P)$ is conservative, (i.e. $\vec{\mathbf{F}}(x,y,z) = \nabla f(x,y,z)$ for some scalar function f), then

$$\int_C \vec{F} \cdot \vec{\mathbf{T}} ds = f(B) - f(A),$$

for any smooth curve C in $\mathcal{R}^3$ taking A to B.

(**b**) Use the product rule for the dot product to prove that if $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$, where $\vec{\mathbf{a}} = \mathbf{r}''(t)$ when $\mathbf{r}(t)$ is the parametrization with respect to time, then

$$\int_C \vec{F} \cdot \vec{\mathbf{T}} ds = \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2.$$

(**c**) Assume further that $\mathbf{F} = m\mathbf{a}$, and $\vec{\mathbf{F}}$ is conservative, so $\vec{\mathbf{F}} = -\nabla P$. Derive the principle of conservation of energy

$$\left\{\frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2\right\} + \{P(B) - P(A)\} = 0.$$

(Hint: Integrate $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$ two different ways.)

Problem #9 (20pts): Let $\vec{\mathbf{F}} = \frac{-y}{r^2}\mathbf{i} + \frac{x}{r^2}\mathbf{j}$ where $r^2 = x^2 + y^2$.

(a) Find $Curl(\vec{\mathbf{F}})$.

(b) Define what a simply connected region is, and use this to explain why the following integral is the same for every positively oriented simple closed curve $\mathcal{C}$ surrounding the origin $(x, y) = (0, 0)$.

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T} ds = ? \tag{2}$$

(c) Evaluate (2) by direct parameterization taking $\mathcal{C}$ to be the unit circle.

(d) Green's Theorem says $\int_{\mathcal{C}} \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds = \int\int_{\mathcal{A}} Curl(\vec{\mathbf{F}}) \cdot \mathbf{k}\, dA$. Explain why Green's Theorem fails in the case of (2).

Problem #10 (20pts): (a) Consider the vector field $\vec{\mathbf{F}} = -\frac{\vec{\mathbf{r}}}{r^3}$ where $r = \sqrt{x^2 + y^2 + z^2}$. (This is Newton's inverse square force field with all constants set equal to one.) We know $\nabla \frac{1}{r} = \vec{\mathbf{F}}$, so $Curl(\vec{\mathbf{F}}) = 0$.

(a) Use $\frac{\partial}{\partial x} r = \frac{x}{r}$, etc, to calculate $Div(\vec{\mathbf{F}})$.

(b) Calculate the flux $\int\int_{\mathcal{S}_{\mathcal{R}}} \vec{\mathbf{F}} \cdot \mathbf{n}\, d\sigma$ where $\mathcal{S}_{\mathcal{R}}$ is a sphere of radius R, (i.e., the surface of a ball of radius R.) You may use that the area of a sphere is $4\pi R^2$.

(c) Write the Divergence Theorem for $\vec{\mathbf{F}}$ and $\mathcal{S}_{\mathcal{R}}$ and explain why it fails for $\vec{\mathbf{F}} = -\frac{\vec{\mathbf{r}}}{r^3}$.