Plane Curves

Find T, N, and κ for the plane curves in Exercises 1–4.

1. \( r(t) = t\mathbf{i} + (\ln \cos t)\mathbf{j}, \quad -\pi/2 < t < \pi/2 \)
2. \( r(t) = (\ln \sec t)\mathbf{i} + t\mathbf{j}, \quad -\pi/2 < t < \pi/2 \)
3. \( r(t) = (2t + 3)\mathbf{i} + (5 - t^2)\mathbf{j} \)
4. \( r(t) = (\cos t + t \sin t)\mathbf{i} + (\sin t - t \cos t)\mathbf{j}, \quad t > 0 \)

5. A formula for the curvature of the graph of a function in the xy-plane
   a. The graph \( y = f(x) \) in the xy-plane automatically has the parametrization \( x = x, y = f(x) \), and the vector formula \( r(x) = x\mathbf{i} + f(x)\mathbf{j} \). Use this formula to show that if \( f \) is a twice-differentiable function of \( x \), then
   \[
   \kappa(x) = \frac{|f''(x)|}{\sqrt{1 + (f'(x))^2}}.
   \]
   b. Use the formula for \( \kappa \) in part (a) to find the curvature of \( y = \ln(\cos x), -\pi/2 < x < \pi/2 \). Compare your answer with the answer in Exercise 1.
   c. Show that the curvature is zero at a point of inflection.

6. A formula for the curvature of a parametrized plane curve
   a. Show that the curvature of a smooth curve \( r(t) = f(t)\mathbf{i} + g(t)\mathbf{j} \) defined by twice-differentiable functions \( x = f(t) \) and \( y = g(t) \) is given by the formula
   \[
   \kappa = \frac{|\ddot{x}y' - \ddot{y}x'|}{(x'^2 + y'^2)^{3/2}}.
   \]
   b. Apply the formula to find the curvatures of the following curves.
      b. \( r(t) = t\mathbf{i} + (\ln \sin t)\mathbf{j}, \quad 0 < t < \pi \)
      c. \( r(t) = [\tan^{-1}(\sinh t)]\mathbf{i} + (\ln \cosh t)\mathbf{j} \)

7. Normals to plane curves
   a. Show that \( n(t) = -g'(t)\mathbf{i} + f'(t)\mathbf{j} \) and \( -n(t) = g'(t)\mathbf{i} - f'(t)\mathbf{j} \) are both normal to the curve \( r(t) = f(t)\mathbf{i} + g(t)\mathbf{j} \) at the point \( (f(t), g(t)) \).
   To obtain \( N \) for a particular plane curve, we can choose the one of \( n \) or \( -n \) from part (a) that points toward the concave side of the curve, and make it into a unit vector. (See Figure 13.21.) Apply this method to find \( N \) for the following curves.
   b. \( r(t) = t\mathbf{i} + e^{2t}\mathbf{j} \)
   c. \( r(t) = \sqrt{4 - t^2} \mathbf{i} + t\mathbf{j}, \quad -2 \leq t \leq 2 \)

8. (Continuation of Exercise 7.)
   a. Use the method of Exercise 7 to find \( N \) for the curve \( r(t) = t\mathbf{i} + (1/3)t^3\mathbf{j} \) when \( t < 0 \); when \( t > 0 \).

b. Calculate
   \[
   \frac{dT}{dt}, \quad t \neq 0,
   \]
   for the curve in part (a). Does \( N \) exist at \( t = 0 \)? Graph the curve and explain what is happening to \( N \) as \( t \) passes from negative to positive values.

Space Curves

Find T, N, and κ for the space curves in Exercises 9–16.

9. \( r(t) = (3 \sin t)\mathbf{i} + (3 \cos t)\mathbf{j} + 4t\mathbf{k} \)
10. \( r(t) = (\cos t + t \sin t)\mathbf{i} + (\sin t - t \cos t)\mathbf{j} + 3\mathbf{k} \)
11. \( r(t) = (e^t \cos t)\mathbf{i} + (e^t \sin t)\mathbf{j} + 2\mathbf{k} \)
12. \( r(t) = (6 \sin 2t)\mathbf{i} + (6 \cos 2t)\mathbf{j} + 5\mathbf{k} \)
13. \( r(t) = (t^3/3)\mathbf{i} + (t^2/2)\mathbf{j}, \quad t > 0 \)
14. \( r(t) = (\cos^3 t)\mathbf{i} + (\sin^3 t)\mathbf{j}, \quad 0 < t < \pi/2 \)
15. \( r(t) = t\mathbf{i} + (a \cosh(t/a))\mathbf{j}, \quad a > 0 \)
16. \( r(t) = (\cosh t)\mathbf{i} - (\sinh t)\mathbf{j} + t\mathbf{k} \)

More on Curvature

17. Show that the parabola \( y = ax^2, a \neq 0 \), has its largest curvature at its vertex and has no minimum curvature. (Note: Since the curvature of a curve remains the same if the curve is translated or rotated, this result is true for any parabola.)

18. Show that the ellipse \( x = a \cos t, y = b \sin t, a > b > 0 \), has its largest curvature on its major axis and its smallest curvature on its minor axis. (As in Exercise 17, the same is true for any ellipse.)

19. Maximizing the curvature of a helix. In Example 5, we found the curvature of the helix \( r(t) = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j} + bt\mathbf{k}, (a, b \neq 0) \), to be \( \kappa = a/(a^2 + b^2) \). What is the largest value \( \kappa \) can have for a given value of \( b \)? Give reasons for your answer.

20. Total curvature. We find the total curvature of the portion of a smooth curve that runs from \( s = s_0 \) to \( s = s_1 \) by integrating \( \kappa \) from \( s_0 \) to \( s_1 \). If the curve has some other parameter, say \( t \), then the total curvature is
   \[
   \kappa = \int_{s_0}^{s_1} \kappa \, ds = \int_{t_0}^{t_1} \kappa \frac{ds}{dt} \, dt = \int_{t_0}^{t_1} \kappa |v| \, dt,
   \]
   where \( t_0 \) and \( t_1 \) correspond to \( s_0 \) and \( s_1 \). Find the total curvatures of
   a. The portion of the helix \( r(t) = (3 \cos t)\mathbf{i} + (3 \sin t)\mathbf{j} + tk, \quad 0 \leq t \leq 4\pi \).
   b. The parabola \( y = x^2, -\infty < x < \infty \).

21. Find an equation for the circle of curvature of the curve \( r(t) = \mathbf{i} + (\sin t)\mathbf{j} \) at the point \((\pi/2, 1)\). (The curve parameterizes the graph of \( y = \sin x \) in the xy-plane.)