15.3 Double Integrals in Polar Form

EXERCISES 15.3

Evaluating Polar Integrals
In Exercises 1–16, change the Cartesian integral into an equivalent polar integral. Then evaluate the polar integral.

1. \( \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \, dx \)
2. \( \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \, dx \)
3. \( \int_{0}^{2\pi} \int_{0}^{r} (x^2 + y^2) \, dx \, dy \)
4. \( \int_{0}^{2\pi} \int_{0}^{r} (x^2 + y^2) \, dx \, dy \)
5. \( \int_{0}^{2\pi} \int_{0}^{r^2} \, dy \, dx \)
6. \( \int_{0}^{2\pi} \int_{0}^{r} \, dx \, dy \)
7. \( \int_{0}^{2\pi} \int_{0}^{r} \, dx \, dy \)
8. \( \int_{0}^{2\pi} \int_{0}^{r} \, dx \, dy \)
9. \( \int_{0}^{2\pi} \int_{0}^{r^{\frac{1}{2}}} \, dy \, dx \)
10. \( \int_{0}^{2\pi} \int_{0}^{r^{\frac{1}{2}}} \, dx \, dy \)
11. \( \int_{0}^{2\pi} \int_{0}^{r} \, dx \, dy \)
12. \( \int_{0}^{2\pi} \int_{0}^{r} \, dx \, dy \)
13. \( \int_{0}^{2\pi} \int_{0}^{r} \, dx \, dy \)
14. \( \int_{0}^{2\pi} \int_{0}^{r} \, dx \, dy \)
15. \( \int_{0}^{2\pi} \int_{0}^{r} \ln(x^2 + y^2 + 1) \, dx \, dy \)
16. \( \int_{0}^{2\pi} \int_{0}^{r} \frac{2}{(1 + x^2 + y^2)^{\frac{3}{2}}} \, dx \, dy \)

Finding Area in Polar Coordinates

17. Find the area of the region cut from the first quadrant by the curve \( r = 2(2 - \sin 2\theta)^{\frac{1}{2}} \).

18. Cardioid overlapping a circle Find the area of the region that lies inside the cardioid \( r = 1 + \cos \theta \) and outside the circle \( r = 1 \).

19. One leaf of a rose Find the area enclosed by one leaf of the rose \( r = 12 \sin 3\theta \).

20. Snail shell Find the area of the region enclosed by the positive \( x \)-axis and spiral \( r = 4\theta/3, 0 \leq \theta \leq 2\pi \). The region looks like a snail shell.

21. Cardioid in the first quadrant Find the area of the region common to the interiors of the cardioids \( r = 1 + \cos \theta \) and \( r = 1 - \cos \theta \).

22. Overlapping cardioids Find the area of the region common to the interiors of the cardioids \( r = 1 + \cos \theta \) and \( r = 1 - \cos \theta \).

Masses and Moments

23. First moment of a plate Find the first moment about the \( x \)-axis of a thin plate of constant density \( \delta(x, y) = 3 \), bounded below by the \( x \)-axis and above by the cardioid \( r = 1 - \cos \theta \).

24. Inertial and polar moments of a disk Find the moment of inertia about the \( x \)-axis and the polar moment of inertia about the origin of a thin disk bounded by the circle \( x^2 + y^2 = a^2 \) if the disk's density at the point \( (x, y) \) is \( \delta(x, y) = k(x^2 + y^2) \), \( k \) a constant.

25. Mass of a plate Find the mass of a thin plate covering the region outside the circle \( r = 3 \) and inside the circle \( r = 6 \sin \theta \) if the plate's density function is \( \delta(x, y) = 1/r \).

26. Polar moment of a cardioid overlapping circle Find the polar moment of inertia about the origin of a thin plate covering the region that lies inside the cardioid \( r = 1 - \cos \theta \) and outside the circle \( r = 1 \) if the plate's density function is \( \delta(x, y) = 1/r^2 \).

27. Centroid of a cardioid region Find the centroid of the region enclosed by the cardioid \( r = 1 + \cos \theta \).

28. Polar moment of a cardioid region Find the polar moment of inertia about the origin of a thin plate enclosed by the cardioid \( r = 1 + \cos \theta \) if the plate's density function is \( \delta(x, y) = 1 \).

Average Values

29. Average height of a hemisphere Find the average height of the hemisphere \( z = \sqrt{a^2 - x^2 - y^2} \) above the disk \( x^2 + y^2 \leq a^2 \) in the \( xy \)-plane.

30. Average height of a cone Find the average height of the (single) cone \( z = \sqrt{x^2 + y^2} \) above the disk \( x^2 + y^2 \leq a^2 \) in the \( xy \)-plane.

31. Average distance from interior of disk to center Find the average distance from a point \( P(x, y) \) in the disk \( x^2 + y^2 \leq a^2 \) to the origin.

32. Average distance squared from a point in a disk to a point in its boundary Find the average value of the square of the distance from the point \( P(x, y) \) in the disk \( x^2 + y^2 \leq 1 \) to the boundary point \( A(1, 0) \).

Theory and Examples

33. Converting to a polar integral Integrate \( f(x, y) = (\ln(x^2 + y^2))/\sqrt{x^2 + y^2} \) over the region \( 1 \leq x^2 + y^2 \leq e \).

34. Converting to a polar integral Integrate \( f(x, y) = (\ln(x^2 + y^2))/(x^2 + y^2) \) over the region \( 1 \leq x^2 + y^2 \leq e^2 \).

35. Volume of noncircular right cylinder The region that lies inside the cardioid \( r = 1 + \cos \theta \) and outside the circle \( r = 1 \) is the base of a solid right cylinder. The top of the cylinder lies in the plane \( z = x \). Find the cylinder's volume.