Name: ______________________

Student ID#: ______________________

Section: ______________________

Midterm Exam 1

Friday, February 2

MAT 21D, Temple, Winter 2018

Show your work on every problem. Correct answers with no supporting work will not receive full credit. Be organized and use notation appropriately. No calculators, notes, books, cellphones, etc. may be used on this exam. Please write legibly. Please have your student ID ready to be checked when you turn in your exam.

Problem	Your Score	Maximum Score
1		20
2		20
3		20
4		20
5		20
Total		100

Problem #1 (20pts): (a) Sketch the region of integration $\mathbf{R}_{xy}$ determined by the iterated integral

$$\int_0^2 \int_{x^3}^{4x} x^2 \sin(xy)\, dydx. \tag{1}$$

Solution: The region above $y = x^3$ and below $y = 4x$ between $0 \leq x \leq 2$.

(b) Rewrite (1) with order of integration reversed. (Do not evaluate)

Solution:

$$\int_0^8 \int_{\frac{1}{4}y}^{y^{1/3}} x^2 \sin(xy)\, dxdy.$$

Problem #2 (20pts): Consider a three dimensional metal object D bounded on the sides by the yz, xz and sy-coordinate planes, and bounded above by the plane that touches the three sides at $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$, where a, b, c are positive numbers. Let the density $\delta = \delta(x, y, z)$ be general. Derive a formula for the three coordinates of the center of mass containing only iterated integrals.

Solution: $\bar{x} = \frac{I_{yz}}{M}$, $\bar{y} = \frac{I_{xz}}{M}$, $\bar{z} = \frac{I_{xy}}{M}$, where

$$M = \int\int\int_D \delta(x, y, z)dA,$$

$$M_{yz} = \int\int\int_D x\delta(x, y, z)dA, \; M_{xz} = \int\int\int_D y\delta(x, y, z)dA, \; M_{xy} = \int\int\int_D z\delta(x, y, z)dA.$$

It remains only to give the iteration of the integrals. For this, the upper plane is $z = -\frac{c}{a}x - \frac{c}{b}y + c$, so the iteration is:

$$\int\int\int_D dA = \int_0^a \int_0^{-\frac{b}{a}x+b} \int_0^{-\frac{c}{a}x-\frac{c}{b}y+c} dzdydx$$

Problem #3a (10pts): Using spherical coordinates, set up the iterated integral for the total mass M of the "ice cream cone" shaped region D cut from the solid sphere $\rho \leq 2$ by the cone $\phi = \pi/3$, assuming the density $\delta(x, y, z) = y$. Set up but do not evaluate.

Solution:

$$\int_0^{2\pi} \int_0^{\pi/3} \int_0^2 (\rho \sin\phi \sin\theta) \left(\rho^2 \sin\phi\right) d\rho d\phi d\theta$$

Problem #3b (10pts): Set up the integral in problem #3 using cylindrical coordinates.

Solution: Top of ice cream cone is $z = \sqrt{4 - r^2}$, bottom is $z = \frac{r}{r \tan \pi/3}$, region of integration in (r, θ) is $0 \leq r \leq 2 \sin \pi/3$, $0 \leq \theta \leq 2\pi$, so

$$\int_0^{2\pi} \int_0^{2 \sin \pi/3} \int_{\frac{r}{\tan \pi/3}}^{\sqrt{4-r^2}} (r \sin \theta)\ dz r dr d\theta$$

Problem #4 (20pts): The mapping $x = u^2 - v^2$, $y = \frac{1}{2}uv$ takes a domain R_{xy} in (x, y)-space to the rectangular region $0 \le u \le 1$, $1 \le v \le 2$. Using the change of coordinates method, (and letting $dA = dxdy$), evaluate

$$\int\int_{R_{xy}} y\, dxdy.$$

Solution:

$$J = Det \begin{vmatrix} 2u & -2v \\ \frac{1}{2}v & \frac{1}{2}u \end{vmatrix} = u^2 + v^2 \tag{2}$$

Thus

$$\begin{aligned}
\int\int_{R_{xy}} y\, dxdy &= \int_1^2 \int_0^1 \frac{1}{2}uv(u^2+v^2)dudv = \frac{1}{2}\int_1^2 \int_0^1 (vu^3 + uv^3)dudv \\
&= \frac{1}{2}\int_1^2 \left[v\frac{u^4}{4} + \frac{u^2}{2}v^3\right]_0^1 dv \\
&= \frac{1}{2}\int_1^2 \left[\frac{v}{4} + \frac{v^3}{2}\right] dv \\
&= \frac{1}{2}\left[\frac{v^2}{8} + \frac{v^4}{8}\right]_1^2 dv \\
&= \frac{1}{16}\left[v^2 + v^4\right]_1^2 \\
&= \frac{1}{16}\left[4 + 16 - 1 - 1\right]_1^2 \\
&= \frac{18}{16} = \frac{9}{8}.
\end{aligned}$$

Problem #5 (20pts):

(a) Use polar coordinates (and the trick in class) to derive the value of $A = \int_{-\infty}^{+\infty} e^{-u^2} du$.

Solution:

$$\int_0^\infty \int_0^\infty e^{-x^2-y^2} dxdy = \int_0^\infty e^{-x^2} dx \int_0^\infty e^{-y^2} dy = \left(\frac{A}{2}\right)^2$$

But

$$\int_0^\infty \int_0^\infty e^{-x^2-y^2} dxdy = \int_0^{\pi/2} \int_0^\infty e^{-r^2} r dr d\theta$$

which, using $u = r^2$, $du = 2rdr$ gives

$$\int_0^\infty \int_0^\infty e^{-x^2-y^2} dxdy = \pi/4 \int_0^\infty e^{-u} du = \pi/4 \left[-e^{-u}\right]_0^\infty = \pi/4.$$

Thus, $\pi/4 = \left(\frac{A}{2}\right)^2$, implies $A = \sqrt{\pi}$.

(b) The Gaussian with mean $\mu > 0$ and variance $\sigma^2 > 0$ is give by $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{\sqrt{2\sigma^2}}}$. Show that $\int_{-\infty}^{+\infty} f(x)dx = 1$ by reducing the case of general μ and σ^2 to the case (a). (σ is called the *standard deviation*.)

Solution: Substitute $u = \frac{(x-\mu)}{\sqrt{2\sigma^2}}$, $du = \frac{dx}{\sqrt{2\sigma^2}}$ gives

$$\int_{-\infty}^{+\infty} f(x)dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-u^2} du = \frac{1}{\sqrt{\pi}} \sqrt{\pi} = 1.$$

Typo in formula for Gaussian— parentheses should cover denominator!

Give almost all credit for right idea, maybe a point or two for catching the mistake…