Name: ______________________

Student ID#: ______________________

Section: ______________________

Midterm Exam 1

Friday, February 1

MAT 21D, Temple, Winter 2019

Print names and ID's clearly, and have your student ID ready to be checked when you turn in your exam. Write the solutions clearly and legibly. Do not write near the edge of the paper or the stapled corner. Show your work on every problem. Correct answers with no supporting work will not receive full credit. Be organized and use notation appropriately. No calculators, notes, books, cellphones, etc. may be used on this exam.

Problem	Your Score	Maximum Score
1		20
2		20
3		20
4		20
5		20
Total		100

Problem #1 (20pts): (a) Sketch the region of integration $\mathbf{R}_{xy}$ determined by the iterated integral

$$\int_0^{\frac{\pi}{2}} \int_{\frac{2x}{\pi}}^{\sin x} f(x, y)\, dy dx. \tag{1}$$

Solution: The region above the line $y = 2x/\pi$ and below $y = \sin x$ for $0 \leq x \leq \frac{\pi}{2}$.

(b) Rewrite (1) with order of integration reversed. (Do not evaluate)

Solution:

$$\int_0^{1} \int_{\sin^{-1} y}^{\frac{\pi y}{2}} f(x, y)\, dx dy.$$

Problem #2 (20pts):
Let R_{xy} denote a two dimensional region in the xy-plane lying within the bounded rectangle $a \le x \le b$, $c \le y \le d$, and recall the definition of a Riemann Sum:

$$\sum_{(x_i,y_j)\in R_{xy}} f(x_i, y_j)\Delta x\Delta y.$$

Draw a picture of R_{xy}, define and label points x_i, y_j assuming $i, j = 1, ..., N$, (including x_0, x_N, y_0, y_N and typical points x_i y_j in between), and give the definition of $\int\int_{R_{xy}} f(x, y)dA$ in terms of the Riemann sums.

Then use this Definition to prove that if both integrals exist, then

$$\int\int_{R_{xy}} f(x, y)dA \le \int\int_{R_{xy}} |f(x, y)|dA$$

DEF: $\int\int_{R_{xy}} f(x, y)dA = \lim_{N\to\infty} \sum_{(x_i,y_j)\in R_{xy}} f(x_i, y_j)\Delta x\Delta y.$

Proof:

$$\sum_{(x_i,y_j)\in R_{xy}} f(x_i, y_j)\Delta x\Delta y \le \sum_{(x_i,y_j)\in R_{xy}} |f(x_i, y_j)|\Delta x\Delta y.$$

so

$$\int\int_{R_{xy}} f(x, y)dA = \lim_{N\to\infty} \sum_{(x_i,y_j)\in R_{xy}} f(x_i, y_j)\Delta x\Delta y \tag{2}$$

$$\le \lim_{N\to\infty} \sum_{(x_i,y_j)\in R_{xy}} |f(x_i, y_j)|\Delta x\Delta y \tag{3}$$

by properties of limits.

Problem #3 (20pts): Consider a two dimensional wheel of radius R_0 meters spinning at ω revolutions per second about the z-axis. (Denote this region R_{xy}.) Assume the density of the wheel is $\delta(r) = ar^4 kg/m^3$. Find a formula for the I_z=Moment of Inertia about the z-axis and the KE=Kinetic Energy of the spinning wheel in terms of R_0, a, ω. (Make sure to give the correct units of both.) Also derive a formula for the radius of gyration R, the radius at which all the mass should be placed to get the same Kinetic Energy.

Solution: Let R_{xy} denote the circle of radius R_0. $KE = \frac{1}{2}I_z\omega^2$ where

$$I_z = \int\int_{R_{xy}} r^2 ar^4 dV = \int_0^{2\pi}\int_0^{R_0} r^2 ar^4 r dr d\theta = 2\pi a\int_0^{R_0} r^7 dr = \frac{2\pi a R_0^8}{8} = \frac{\pi a R_0^8}{4}.$$

Dimensions of KE are $[KE] = \frac{kg\, m^2}{s^2}$, $[\omega] = 1/s$, so $[I_z] = kg\, m^2$.

Radius of gyration R satisfies $\frac{1}{2}MR^2\omega^2 = \frac{1}{2}I_z\omega^2$ so $R = \sqrt{I_z M}$, where $M = \int\int_{R_{xy}} ar^4 k dA$.

Problem #4a (10pts): Using spherical coordinates, set up the iterated integral for the total mass M of the region D cut from the solid sphere $\rho \leq a$ by restricting $0 \leq \phi \leq b \leq \pi$, assuming $0 < b < \pi$, and the density $\delta(x, y, z) = xyz^2$. Set up but do not evaluate.

Solution:

$$\int_0^{2\pi} \int_0^b \int_0^a (\rho \sin\phi \cos\theta)(\rho \sin\phi \sin\theta)(\rho \cos\phi)^2 \left(\rho^2 \sin\phi\right) d\rho d\phi d\theta$$

Problem #4b (10pts): Set up the integral in problem #5a using cylindrical coordinates in the case $b = \pi/4$.

Solution: Top of ice cream cone is $z^2 + r^2 = b^2$, bottom is $z = r$, region of integration in (r, θ) is $0 \le r \le b/\sqrt{2}$, $0 \le \theta \le 2\pi$, so

$$\int_0^{2\pi} \int_0^{b/\sqrt{2}} \int_r^{\sqrt{z^2-r^2}} (r\cos\theta)(r\sin\theta)(z^2) r dz dr d\theta$$

Problem #5 (20pts): The mapping $x = u^2 - v^2$, $y = \frac{a}{2}uv$ takes a domain R_{xy} in (x, y)-space to the rectangular region $0 \le u \le 1$, $a \le v \le 2a$, for $a > 0$. Using the change of coordinates method, find a formula for

$$\int\int_{R_{xy}} x\, dA_{xy}.$$

as a function of a.

Solution:

$$J = Det \begin{vmatrix} 2u & -2v \\ \frac{a}{2}v & \frac{a}{2}u \end{vmatrix} = a(u^2 + v^2) \tag{4}$$

Thus

$$\begin{aligned}
\int\int_{R_{xy}} y\, dxdy &= \int_a^{2a}\int_0^1 a(u^2 - v^2)(u^2 + v^2)dudv = a\int_a^{2a}\int_0^1 (u^4 - v^4)dudv \\
&= a\int_a^{2a} \left[\frac{u^5}{5} - v^4 u\right]_{u=0}^{u=1} dv \\
&= a\int_a^{2a} \left[\frac{1}{5} - v^4\right] dv \\
&= \frac{a^2}{5} - \left[\frac{v^5}{5}\right]_a^{2a} dv \\
&= \frac{a^2}{5} - \frac{a^5}{5}\left[2^5 - 1\right].
\end{aligned}$$

Extra Credit: (5pts) Use polar coordinates (and the trick in class) to derive the value of $A = \int_0^{+\infty} e^{-u^2} du$ by use of properties of double integrals.

Solution:

$$\int_0^\infty \int_0^\infty e^{-x^2-y^2} dxdy = \int_0^\infty e^{-x^2} dx \int_0^\infty e^{-y^2} dy = A^2$$

But

$$\int_0^\infty \int_0^\infty e^{-x^2-y^2} dxdy = \int_0^{\pi/2} \int_0^\infty e^{-r^2} r dr d\theta$$

which, using $u = r^2$, $du = 2rdr$ gives

$$\int_0^\infty \int_0^\infty e^{-x^2-y^2} dxdy = \pi/4 \int_0^\infty e^{-u} du = \pi/4 \left[-e^{-u}\right]_0^\infty = \pi/4.$$

Thus, $\pi/4 = A^2$, implies $A = \sqrt{\pi}/2$.