Name: ______________________

Student ID#: ______________________

Section: ______________________

Midterm Exam 2

Wednesday, March 6

MAT 21D, Temple, Winter 2019

Print names and ID's clearly, and have your student ID ready to be checked when you turn in your exam. Write the **Solution**s clearly and legibly. Do not write near the edge of the paper or the stapled corner. Show your work on every problem. Correct answers with no supporting work will not receive full credit. Be organized and use notation appropriately. No calculators, notes, books, cellphones, etc. may be used on this exam.

Problem	Your Score	Maximum Score
1		20
2		20
3		20
4		20
5		20
Total		100

Problem #1 (20pts): Assume that a particle at position $\mathbf{r}(t)$ is moving in the plane, in circular motion around the origin, starting at $\mathbf{r}(0) = (1, 0)$. Assume that it is accelerating, with a speed increasing by $ds/dt = e^t$. Find a formula for the position $\mathbf{r}(t) = \overrightarrow{(x(t), y(t))}$ at time t, $-\infty < t < +\infty$. Use this to find the velocity vector $\mathbf{v}(t)$, the unit tangent vector $\mathbf{T}(t)$, the acceleration vector $\mathbf{a}(t)$, the principle normal N, and the curvature κ.

Solution: Define $\vec{r}(t) = \cos(e^t - 1)\mathbf{i} + \sin(e^t - 1)\mathbf{j}$. Then

$$\vec{r}\,'(t) = \vec{v}(t) = e^t \left\{ -\sin(e^t - 1)\mathbf{i} + \cos(e^t - 1)\mathbf{j} \right\},$$

so

$$ds/dt = \|\vec{v}(t)\| = e^t$$

as required. Then

$$\vec{T}(t) = \cos(e^t - 1)\mathbf{i} + \sin(e^t - 1)\mathbf{j},$$

and

$$\vec{r}\,''(t) = \vec{a}(t) = e^2 t \left\{ -\cos(e^t - 1)\mathbf{i} - \sin(e^t - 1)\mathbf{j} \right\}.$$

Finally, circular motion around unit circle implies

$$\kappa = 1/r = 1,$$

and

$$\vec{\mathbf{N}} = -\cos(e^t - 1)\mathbf{i} - \sin(e^t - 1)\mathbf{j}.$$

Problem #2 (20pts): Let

$$\overrightarrow{\mathbf{F}} = \overrightarrow{(2y^2, 4xy + z, y + \cos(z))}$$

(**a**) Show that $Curl(\overrightarrow{\mathbf{F}}) = 0$.

Solution:

$$\begin{aligned} Curl(\overrightarrow{\mathbf{F}}) &= \mathbf{i}([y + \cos(z)]_y - [4xy + z]_z) - \mathbf{j}([y + \cos(z)]_x - [2y^2]_z) \\ &\quad + \mathbf{k}([4xy + z]_x - [2y^2]_y) = 0 \end{aligned}$$

(**b**) Use the method of partial integration to find f such that $\overrightarrow{\mathbf{F}} = \nabla f$.

Solution: We find f such that $\nabla f = \overrightarrow{F}$:

$$\begin{aligned}
f &= \int_x 2y^2 = 2y^2 x + g(y,z) \\
\frac{\partial f}{\partial y} &= 4yx + \frac{\partial g}{\partial y} = 4xy + z \\
g &= zy + h(z) \\
\frac{\partial f}{\partial z} &= y + h'(z) = y + \cos z \\
h &= \sin z
\end{aligned}$$

so

$$f(x,y,z) = 2xy^2 + yz + \sin z$$

Problem #3 (20pts): Recall the chain rule for functions of two variables:

$$\frac{d}{dt}f(x(t), y(t)) = f_x\dot{x} + f_y\dot{y}.$$

(**a**) Use the chain rule to prove that if a vector field $\overrightarrow{\mathbf{F}} = \overrightarrow{(M, N)}$ is conservative, (i.e. $\overrightarrow{\mathbf{F}}(x, y) = \nabla f(x, y)$ for some scalar function f), then

$$\int_C \overrightarrow{F} \cdot \overrightarrow{\mathbf{T}} ds = f(B) - f(A),$$

for any smooth curve C in $\mathcal{R}^2$ taking A to B.

Solution: Let $\overrightarrow{\mathbf{r}}(t))$ be a parameterization of C, $t_A \leq t \leq t_B$, $\overrightarrow{\mathbf{r}}(t_A)) = A$, $\overrightarrow{\mathbf{r}}(t_B)) = B$.

$$\begin{aligned}
\int_C \overrightarrow{F} \cdot \overrightarrow{\mathbf{T}} ds &= \int_{t_A}^{t_B} \overrightarrow{\nabla f} \cdot \overrightarrow{\mathbf{v}} dt \\
&= \int_{t_A}^{t_B} \frac{d}{dt} f(\overrightarrow{\mathbf{r}}(t)) dt \\
&= \overrightarrow{\mathbf{r}}(t_B)) - \overrightarrow{\mathbf{r}}(t_A)) = B - A
\end{aligned}$$

(**b**) Use the product rule for the dot product to prove that if $\overrightarrow{\mathbf{F}} = m\overrightarrow{\mathbf{a}}$, where $\overrightarrow{\mathbf{a}} = \mathbf{r}''(t)$ when $\mathbf{r}(t)$ is the parametrization with respect to time, then

$$\int_C \overrightarrow{F} \cdot \overrightarrow{\mathbf{T}} ds = \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2.$$

Solution:

$$\begin{aligned}
\int_C \overrightarrow{F} \cdot \overrightarrow{\mathbf{T}} ds &= \int_{t_A}^{t_B} \overrightarrow{F} \cdot \overrightarrow{\mathbf{v}} dt \\
&= \int_{t_A}^{t_B} \overrightarrow{a} \cdot \overrightarrow{\mathbf{v}} dt \\
&= \int_{t_A}^{t_B} \frac{d}{dt}\frac{1}{2}\overrightarrow{\mathbf{v}} \cdot \overrightarrow{\mathbf{v}} dt \\
&= \frac{1}{2}\|\overrightarrow{\mathbf{v_B}}\|^2 - \frac{1}{2}\|\overrightarrow{\mathbf{v_A}}\|^2
\end{aligned}$$

Problem #4 (20pts): Green's Theorem states that

$$\int_C \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{T}} ds = \int\int_A N_x - M_y \, dA,$$

where A is the region in the xy-plane bounded by positively oriented closed curve C. Verify Green's theorem in the case when $\overrightarrow{\mathbf{F}} = xy\overrightarrow{\langle 1,2\rangle}$, and C is the boundary of the triangular region bounded by the x-axis, y-axis and the line $y = -2x + 2$. Draw a picture.

Solution: Since $\overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{T}} = 0$ along the x-axis and y-axis, we need only evaluate the LHS along the line $y = -2x + 2$. Letting $\overrightarrow{\mathbf{r}}(t) = (t, -2t + 2)$ gives a negative parameterization of this straight line curve. Thus for the LHS:

$$\begin{aligned}
\int_C \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{T}} ds &= -\int_0^1 \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{v}} \, dt \\
&= -\int_0^1 t(-2t+2)\overrightarrow{\langle 1,2\rangle} \cdot \overrightarrow{\langle 1,-2\rangle} \, dt \\
&= 3\int_0^1 -2t^2 + 2t \, dt \\
&= 3[-\frac{2}{3}t^3 + t^2]_0^1 = -2 + 3 = 1.
\end{aligned}$$

For the RHS:

$$\begin{aligned}
\int\int_A N_x - M_y \, dA &= \int_0^1 \int_0^{-2x+2} 2y - x \, dy dx \\
&= \int_0^1 y^2 - xy]_{y=0}^{y=-2x+2} \, dx \\
&= \int_0^1 (-2x+2)^2 - x(-2x+2) \, dx \\
&= \int_0^1 4x^2 - 8x + 4 + 2x^2 - 2x \, dx \\
&= \int_0^1 6x^2 - 10x + 4 \, dx \\
&= 2x^3 - 5x^2 + 4x]_0^1 = 2 - 5 + 4 = 1.
\end{aligned}$$

Problem #5 (20pts): Consider Kepler's Laws under the simplifying assumption that the planets move in circular orbits with the sun at the center, (not a terrible approximation). In this case, each planet moves around a circle with position vector

$$\vec{\mathbf{r}}(t) = R\cos\omega t\,\mathbf{i} + R\sin\omega t\,\mathbf{j}, \tag{1}$$

where R and ω are constants which depend on the planet, and t is the time. Assuming Newton's inverse square force law in the form

$$\overrightarrow{\mathbf{a}} = -G\frac{1}{r^2}\frac{\overrightarrow{\mathbf{r}}}{r}, \tag{2}$$

(where G is Newton's gravitational constant), derive Kepler's third law

$$\frac{T^2}{R^3} = K, \tag{3}$$

where T is the period of the planet's rotation and K is a constant independent of the planet.

Solution: Differentiating (1) twice gives

$$\vec{\mathbf{a}} = \vec{\mathbf{r}}''(t) = -\omega^2\vec{\mathbf{r}},$$

and setting this equal to $\overrightarrow{\mathbf{a}}$ in (2) gives

$$-\omega^2 = -\frac{G}{r^3} = -\frac{G}{R^3}, \tag{4}$$

where we use that $r = R$ on the circle of radius R. Now the period T of (1) satisfies $\omega T = 2\pi$, so $T = 2\pi/\omega$ and $\omega^2 = 4\pi^2/T^2$. Putting this into (4) gives

$$\frac{4\pi^2}{T^2} = \frac{G}{R^3},$$

or

$$\frac{T^2}{R^3} = \frac{4\pi^2}{G},$$

where the right hand side is independent of the planet.