Name: Solutions

Student ID#: ______________________

Section: ______________________

Final Exam

Monday June 6

MAT 21D, Temple, Spring 2011

Show your work on every problem. Correct answers with no supporting work will not receive full credit. Be organized and use notation appropriately. No calculators, notes, books, cellphones, etc. Please write legibly. Please have your student ID ready to be checked when you turn in your exam.

Problem	Your Score	Maximum Score
1		25
2		25
3		25
4		25
5		25
6		25
7		25
8		25
Total		200

Problem #1: Let

$$\mathbf{F}(x, y, z) = y \cos(xy)\,\mathbf{i} + \{x \cos(xy) + z\}\,\mathbf{j} + y\,\mathbf{k}.$$

Use the systematic method to find f such that $\mathbf{F} = \nabla f$.

$$f(x,y,z) = \int_x y\cos(xy)\,dy \underset{\substack{\uparrow \\ u = xy \\ du = y\,dx}}{=} \sin(xy) + g(y,z)$$

$$f_y = x\cos(xy) + g_y = x\cos(xy) + z \Rightarrow g_y = z$$

$$g = zy + h(z)$$

$$f_z = g_z = y + h'(z) = y \Rightarrow h'(z) = 0$$

Conclude: $\boxed{f(x,y,z) = \sin(xy) + yz}$

Problem #2: Let $\mathbf{F}(x, y, z) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$ where $f(x, y) = x^2y + y$. Find $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$ where C is the *hypo-elliptic-flexical-elastocele*, a smooth curve that starts at $P = (2, 1, 1)$ and ends at $Q = (-1, 1, 0)$.

$$\int_C \vec{F} \cdot \vec{T}\, ds = \int_C \nabla f \cdot \vec{T}\, ds = f(Q) - f(P)$$

$$= (-1)^2 \cdot (1) + 0 - 2^2 \cdot 1 + 1$$

$$= 1 - 3 = \boxed{-2}$$

Problem #3: Let $\mathbf{F} = x^2\mathbf{i} + xy\mathbf{j} + y\mathbf{k}$.

(a) Compute $Curl\,\mathbf{F}$.

$$\mathrm{Curl}\,\vec{F} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \partial_x & \partial_y & \partial_z \\ x^2 & xy & y \end{vmatrix} = \underset{\sim}{i}(1-0) - \underset{\sim}{j}(0-0) + \underset{\sim}{k}(y-0)$$

$$= \underset{\sim}{i} + y\,\underset{\sim}{k}$$

(b) Find the circulation per area of $\mathbf{F}$ around the z-axis at the point $(x, y, z) = (1, -2, -1)$.

$$\mathrm{Curl}\,\vec{F} \cdot \underset{\sim}{k} = \overrightarrow{(1, 0, y)} \cdot \overrightarrow{(0, 0, 1)} = y$$

$$y = -2 \Rightarrow \mathrm{Curl}\,\vec{F} \cdot k \Big|_{(1,-2,-1)} = -2$$

(c) Find the axis around which the circulation of $\mathbf{F}$ is maximal at the point $(x, y, z) = (1, -2, -1)$.

"Axis of Curl" $= \underset{\sim}{i} + y\,\underset{\sim}{k}$

(d) Find the maximum circulation per area of $\mathbf{F}$ at the point $(x, y, z) = (1, -2, -1)$.

$$\|\mathrm{Curl}\,\vec{F}(1,-2,-1)\| = \sqrt{1^2 + y^2} = \sqrt{5}$$

$y = -2$

Problem #4: Let $\mathcal{S}$ be the surface defined by

$$\begin{aligned} x &= u+v \\ y &= uv \\ z &= v \end{aligned}$$

as (u,v) ranges over $0 \le u \le a,\ \ 0 \le v \le b$. Evaluate the flux $\int\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{n}\,d\sigma$ of F through $\mathcal{S}$ in the direction of positive y, where $\mathbf{F} = y\mathbf{i} + x\mathbf{j}$.

$\vec{r} = (u+v)\underset{\sim}{i} + uv\underset{\sim}{j} + v\underset{\sim}{k}$, $\vec{r}_u = \underset{\sim}{i} + v\underset{\sim}{j}$, $\vec{r}_v = \underset{\sim}{i} + u\underset{\sim}{j} + \underset{\sim}{k}$

$$\text{Flux} = \iint_{\mathcal{S}} \vec{F}\cdot\vec{n}\,d\sigma = \int_0^b\int_0^a \overrightarrow{(y,x,0)}\cdot\vec{n}\ \|\vec{r}_u\times\vec{r}_v\|\,du\,dv$$

$$\vec{r}_u\times\vec{r}_v = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & v & 0 \\ 1 & u & 1 \end{vmatrix} = v\underset{\sim}{i} - \underset{\sim}{j} + (u-v)\underset{\sim}{k}$$

$$\vec{n} = -\frac{\vec{r}_u\times\vec{r}_v}{\|\vec{r}_u\times\vec{r}_v\|} = \frac{(-v,1,u-v)}{\|\vec{r}_u\times\vec{r}_v\|} \quad (\text{pos } y\,!)$$

$$\text{Flux} = \int_0^b\int_0^a \overrightarrow{(uv,u+v,0)}\cdot\frac{\overrightarrow{(-v,1,u-v)}}{\cancel{\|\vec{r}_u\times\vec{r}_v\|}}\,\cancel{\|\vec{r}_u\times\vec{r}_v\|}\,du\,dv$$

$$= \int_0^b\int_0^a -uv^2+u+v\,du\,dv = \int_0^b -\frac{u^2v^2}{2}+\frac{u^2}{2}+vu\Big]_{u=0}^{u=a}dv$$

$$= \int_0^b -\frac{a^2v^2}{2}+\frac{a^2}{2}+av\,dv = -\frac{a^2v^3}{6}+\frac{a^2v}{2}+\frac{av^2}{2}\Big]_0^b = -\frac{a^2b^3}{6}+\frac{a^2b}{2}+\frac{ab^2}{2}$$

Problem #5: Show that if $\mathbf{F} = -\frac{y}{2}\,\mathbf{i} + \frac{x}{2}\,\mathbf{j}$, then for any closed curve C in the (x, y)-plane, $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$=Area enclosed by C. [Hint: Green's Theorem]

$$\int_C \vec{F}\cdot\vec{T}\,ds = \iint_A N_x - M_y\,dx\,dy$$

$$= \iint_A \frac{\partial}{\partial x}\left(\frac{x}{2}\right) - \left(\frac{\partial}{\partial y}\left(-\frac{y}{2}\right)\right) dx\,dy$$

$$= \iint_A \left(\frac{1}{2} + \frac{1}{2}\right) dx\,dy$$

$$= \text{Area } A$$

Problem #6: In words, the value of the divergence of a vector valued function $\mathbf{F}$ at a point $\mathbf{x_0} = (x_0, y_0, z_0)$ is "the flux of $\mathbf{F}$ per volume" at $\mathbf{x_0}$. State the Divergence Theorem and use it to give a careful derivation of this physical interpretation of *Div* $\mathbf{F}$.

Let $B_\varepsilon(\underset{\sim}{x}_0)$ be the ball of center $\underset{\sim}{x}_0$ & radius ε. By Div Thm

$$\iiint_{B_\varepsilon(\underset{\sim}{x}_0)} \operatorname{div}\vec{F}\, dV = \iint_{\partial B_\varepsilon(\underset{\sim}{x}_0)} \vec{F}\cdot\vec{n}\, d\sigma$$

For ε suff small, $\operatorname{div}\vec{F} \approx \operatorname{div}\vec{F}(\underset{\sim}{x}_0)$ so

$$\operatorname{div}\vec{F}(\underset{\sim}{x}_0) \iiint_{B_\varepsilon(\underset{\sim}{x}_0)} dV \approx \iint_{\partial B_\varepsilon(\underset{\sim}{x}_0)} \vec{F}\cdot\vec{n}\, d\sigma$$

So:

$$\operatorname{div}\vec{F}(\underset{\sim}{x}_0) = \lim_{\varepsilon\to 0} \frac{1}{\operatorname{Vol} B_\varepsilon} \iint_{\partial B_\varepsilon(\underset{\sim}{x}_0)} \vec{F}\cdot\vec{n}\, d\sigma$$

= "Flux per volume at $\underset{\sim}{x}_0$"

Problem #7: The principle of *conservation of mass* states that the time rate of change of total mass inside a volume $\mathcal{V}$ is equal to the flux of mass out through the boundary $\partial\mathcal{V}$. Use the Divergence Theorem to show that if a moving density $\rho(\mathbf{x}, t)$ and velocity $\mathbf{v}(\mathbf{x}, t)$ satisfies the continuity equation $\rho_t + Div(\rho\mathbf{v}) = 0$, then the principle of *conservation of mass* holds for every volume $\mathcal{V}$.

Assuming $\rho_t + Div(\rho\vec{v}) = 0$ & integrating ove $\mathcal{V}$ gives

$$\iiint_{\mathcal{V}} \rho_t + Div(\rho\vec{v})\, dv = 0$$

$$\underbrace{\frac{d}{dt}\iiint_{\mathcal{V}} \rho\, dv}_{\text{Total mass in } \mathcal{V}} = -\iiint_{\mathcal{V}} Div(\rho\vec{v})\, dv = \underbrace{-\iint_{\partial\mathcal{V}} \rho\vec{v}\cdot\hat{n}\, d\sigma}_{\text{minus the flux of mass out thru } \partial\mathcal{V}}$$

Problem #8: When the sun is taken as the origin, a planet moving in planar motion around the sun *sweeps out equal area in equal time* if its motion $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ satisfies

$$r^2\dot{\theta} = H = constant.$$

Show that if a planet *sweeps out equal area in equal time*, (as Kepler knew in 1609), then the planet's acceleration vector $\mathbf{a} = \ddot{\mathbf{r}}(t)$ points in the direction from the planet to the sun, (as Newton proved in 1687). Said differently, since force equals mass times acceleration, the force is coming FROM the SUN. (EUREKA!!! The sun, NOT the earth, is the CENTER of the UNIVERSE!!!)

Change to polar coords: $\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$

$\dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$

$$\ddot{x} = \ddot{r}\cos\theta - 2\dot{r}\dot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta - r\ddot{\theta}\sin\theta \quad (1)$$

$$\ddot{y} = \ddot{r}\sin\theta + 2\dot{r}\dot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta + r\ddot{\theta}\cos\theta \quad (2)$$

mult (1) by $\cos\theta$ & (2) by $\sin\theta$ & add:

$$\ddot{x}\cos\theta + \ddot{y}\sin\theta = \ddot{r} - r\dot{\theta}^2 = \ddot{r} - \frac{H^2}{r^3} \quad (3)$$

mult (1) by $\sin\theta$ & (2) by $\cos\theta$ & subtract:

$$-\ddot{x}\sin\theta + \ddot{y}\cos\theta = 2\dot{r}\dot{\theta} + r\ddot{\theta} = 0 \quad (4)$$

Since

$r^2\dot{\theta} = H \Rightarrow 2\dot{r}\dot{\theta} + r\ddot{\theta} = 0$

Now mult (3) by $\cos\theta$ & (4) by $\sin\theta$ & subtr

$$\ddot{x} = \left(\ddot{r} - \frac{H^2}{r^3}\right)\cos\theta = \frac{\ddot{r} - H^2/r^3}{r} x$$

Mult (3) by $\sin\theta$ & (4) by $\cos\theta$ & add

$$\ddot{y} = \left(\ddot{r} - \frac{H^2}{r^3}\right)\sin\theta = \frac{\ddot{r} - H^2/r^3}{r} y$$

Conclude -

$$\vec{a} = \overrightarrow{(\ddot{x}, \ddot{y})} = \frac{\ddot{r} - H^2/r^3}{r} \overrightarrow{(x, y)}$$

$$= \left(\ddot{r} - H^2/r^3\right) \frac{\vec{r}}{\|\vec{r}\|}$$

as claimed !